

# STRICHARTZ ESTIMATES FOR WAVE EQUATIONS WITH CHARGE TRANSFER HAMILTONIAN

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**ABSTRACT.** We prove Strichartz estimates (both regular and reversed) for a scattering state to the wave equation with a charge transfer Hamiltonian in  $\mathbb{R}^3$ :

$$\partial_{tt}u - \Delta u + \sum_{j=1}^m V_j(x - \vec{v}_j t) u = 0.$$

The energy estimate and the local energy decay of a scattering state are also established. In order to study nonlinear multisoliton systems, we will present the inhomogeneous generalizations of Strichartz estimates. As an application of our results, we show that scattering states indeed scatter to solutions to the free wave equation.

## 1. INTRODUCTION

In this paper, we study wave equations with charge transfer Hamiltonian in  $\mathbb{R}^3$ . To be more precise, consider the wave equation with the time-dependent charge transfer Hamiltonian

$$(1.1) \quad H(t) = -\Delta + \sum_{j=1}^m V_j(x - \vec{v}_j t)$$

where  $V_j(x)$ 's are rapidly decaying smooth potentials and  $\{\vec{v}_j\}$  is a set of distinct constant velocities such that

$$(1.2) \quad |\vec{v}_i| < 1, 1 \leq i \leq m.$$

Due to the nature of our problem, we focus on initial data in the energy space. We will prove Strichartz estimates, energy estimates, the local energy decay and the boundedness of the total energy for a scattering state to the wave equation

$$(1.3) \quad \partial_{tt}\psi + H(t)\psi = 0$$

associated with a charge transfer Hamiltonian  $H(t)$ . Throughout, we use  $\partial_{tt}u := \frac{\partial^2}{\partial t \partial t}$ ,  $u_t := \frac{\partial}{\partial t}u$ ,  $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i \partial x_i}$  and occasionally,  $\square := -\partial_{tt} + \Delta$ .

**1.1. Historical background.** In this subsection, we briefly discuss some background of Strichartz estimates, reversed Strichartz estimates.

Our starting point is the free wave equation ( $H_0 = -\Delta$ ) on  $\mathbb{R}^3$

$$(1.4) \quad \partial_{tt}u - \Delta u = 0$$

with initial data

$$(1.5) \quad u(x, 0) = g(x), u_t(x, 0) = f(x).$$

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We can write down  $u$  explicitly,

$$(1.6) \quad u = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f + \cos(t\sqrt{-\Delta}) g.$$

Then for  $p > \frac{2}{s}$  and  $(p, q)$  satisfying

$$(1.7) \quad \frac{3}{2} - s = \frac{1}{p} + \frac{3}{q},$$

one has

$$(1.8) \quad \|u\|_{L_t^p L_x^q} \lesssim \|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s-1}}.$$

Strichartz estimates (1.8), which are stated precisely in Theorem 2.1, are estimates of solutions in terms of space-time integrability properties. The non-endpoint estimates for the wave equations can be found in Ginibre-Velo [GV]. Keel–Tao [KT] also obtained sharp Strichartz estimates for the free wave equation in  $\mathbb{R}^n$ ,  $n \geq 4$  and everything except the endpoint in  $\mathbb{R}^3$ . See Keel–Tao [KT] and Tao’s book [Tao] for more details on the subject’s background and the history.

In  $\mathbb{R}^3$ , there is no hope to obtain such an estimate with the  $L_t^2 L_x^\infty$  norm, the so-called endpoint Strichartz estimate for free wave equations, cf. Klainerman-Machedon [KM] and Machihara-Nakamura-Nakanishi-Ozawa [MNNO]. But if we reverse the order of space-time integration, one can obtain a version of reversed Strichartz estimates from the Morawetz estimate, cf. Theorem 2.2:

$$(1.9) \quad \left\| \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2(\mathbb{R}^3)},$$

$$(1.10) \quad \left\| \cos(t\sqrt{-\Delta}) g \right\|_{L_x^\infty L_t^2} \lesssim \|g\|_{\dot{H}^1(\mathbb{R}^3)}.$$

These types of estimates are extended to inhomogeneous cases and perturbed Hamiltonians in Goldberg-Beceanu [BecGo]. In Section 2 and Section 3, we will rely crucially on these estimates and their generalizations.

Consider a linear wave equation with a real-valued stationary potential in  $\mathbb{R}^3$ ,

$$(1.11) \quad H = -\Delta + V,$$

$$(1.12) \quad \partial_{tt}u + Hu = \partial_{tt}u - \Delta u + Vu = 0,$$

$$(1.13) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

One substantial difference between the perturbed Hamiltonian  $H = -\Delta + V$  and the free Laplacian  $-\Delta$  is the possible existence of eigenvalues and bound states, i.e.,  $L^2$  eigenfunctions of  $H$ . For the class of short-range potentials we consider in this paper, the essential spectrum of  $H$  is  $[0, \infty)$  and the point spectrum may include a countable number of eigenvalues in a bounded subset of the real axis that is discrete away from zero. We further assume that zero is a regular point of the spectrum of  $H$ . Under our hypotheses  $H$  only has pure absolutely continuous spectrum on  $[0, \infty)$  and a finite number of negative eigenvalues. It is very crucial to notice that if  $E < 0$  is a negative eigenvalue, the associated eigenfunction responds to the wave equation propagators by scalar multiplication by  $\cos(t\sqrt{E})$  or  $\frac{\sin(t\sqrt{E})}{E^{\frac{1}{2}}}$ , both of which will grow exponentially since  $\sqrt{E}$  is purely imaginary. Thus, dispersive estimates and Strichartz estimates for  $H$  must include a projection  $P_c$  onto the continuous spectrum in order to get away from this situation.

The problem of dispersive decay and Strichartz estimates for the wave equation with a potential has received much attention in recent years, see the papers by Beceanu-Goldberg [BecGo], Krieger-Schlag [KS] and the survey by Schlag [Sch] for further details and references.

The Strichartz estimates in this case are in the form:

$$(1.14) \quad \left\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f + \cos(t\sqrt{H}) P_c g \right\|_{L_t^p L_x^q} \lesssim \|g\|_{\dot{H}^1} + \|f\|_{L^2},$$

with  $2 < p$ ,  $\frac{1}{2} = \frac{1}{p} + \frac{3}{q}$ . One also has the endpoint reversed Strichartz estimates:

$$(1.15) \quad \left\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f + \cos(t\sqrt{H}) P_c g \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1},$$

see Theorem 2.3.

There are extra difficulties when dealing with time-dependent potentials. For example, given a general time-dependent potential  $V(x, t)$ , it is not clear how to introduce an analog of bound states and a spectral projection. The evolution might not satisfy group properties any more. It might also result in the growth of certain norms of the solutions, see the book by Bourgain [Bou]. In this paper, we focus on the charge transfer Hamiltonian (1.1) in  $\mathbb{R}^3$ :

$$(1.16) \quad H(t) = -\Delta + \sum_{j=1}^m V_j(x - \vec{v}_j t),$$

which appears naturally in the study of nonlinear multisoliton system, see Rodnianski-Schlag-Soffer [RSS2] for the Schrödinger model. For the wave model,

$$(1.17) \quad \partial_{tt} u - \Delta u + \sum_{j=1}^m V_j(x - \vec{v}_j t) u = 0, \quad |\vec{v}_i| < 1, 1 \leq i \leq m,$$

in this paper, we prove Strichartz estimates, energy estimates, the local energy decay which are essential to analyze the stability of noninteracting multi-soliton states.

The study of Schrödinger equations with a charge transfer Hamiltonian can be found in Rodnianski-Schlag-Soffer [RSS], Cai [Cai], Chen [GC1] and Deng-Soffer-Yao [DSY]. For the Schrödinger model, there is no need to require  $|\vec{v}_i| < 1$ . In Rodnianski-Schlag-Soffer [RSS], the authors proved the dispersive estimates for both the scalar and matrix Schrödinger charge transfer models. They introduced Galilei transformations to interchange stationary frames with respect to different potentials. Basically, they applied a bootstrap argument via a semi-classical propagation lemma for low frequencies and Kato's smoothing estimate for high frequencies. With careful analysis of wave operators, the authors also obtain the results on the asymptotic completeness. Their works inspired the subsequent development in Cai [Cai] where the  $L^1 \rightarrow L^\infty$  dispersive estimate is proved. Later on, by Chen [GC1], Strichartz estimates for both the scalar and matrix Schrödinger charge transfer models were presented based on a time-dependent local decay estimate and the endpoint Strichartz estimate for the free equations. Alternatively, Strichartz estimates can be obtained by analysis of wave operators, see Deng-Soffer-Yao [DSY].

Compared with Schrödinger equations, wave equations have some natural difficulties, for example the evolution of bound states of wave equations leads to exponential growth as we pointed out above, meanwhile the evolution of bound states of Schrödinger equations are merely multiplied by oscillating factors. The structure of wave operators in the wave equation setting is not clear either. Moreover, the endpoint Strichartz estimate for free equations, an important tool used in the paper [GC1], also fails for free wave equations in  $\mathbb{R}^3$ . Last but not least, Lorentz transformations are space-time rotations, therefore one can not hope to succeed by the approach used with Schrödinger equations based on Galilei transformations. Galilei transformations are bounded in any  $L^p$  space, but it is not clear under Lorentz transformations whether the energy with respect to the new frame stays comparable to the energy in the original frame. To the author's knowledge, for wave equations with even just one potential moving along a space-like line, Strichartz estimates, scattering, and the asymptotic decomposition of the evolution are new. We refer to [GC2] for more information on wave equations with one moving potential.

**1.2. Charge transfer model and main results.** Before we give the precise definition of our model, it is necessary to introduce Lorentz transformations. Given a vector  $\vec{\mu} \in \mathbb{R}^3$ , there is a Lorentz transformation  $L(\vec{\mu})$  acting on  $(x, t) \in \mathbb{R}^{3+1}$  such that it makes the moving frame  $(x - \vec{\mu}t, t)$  stationary. We can use a  $4 \times 4$  matrix  $B(\vec{\mu})$  to represent the transformation  $L(\vec{\mu})$ . Moreover, for the given vector  $\vec{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$ , there is a  $3 \times 4$  matrix  $M(\vec{\mu})$  such that

$$(1.18) \quad (x - \vec{\mu}t)^T = M(\vec{\mu})(x, t)^T,$$

where the superscript  $T$  denotes the transpose of a vector.

With the preparations above, we can set up our model. We consider the scalar charge transfer model for wave equations in the following sense:

**Definition 1.1.** By a wave equation with a charge transfer Hamiltonian we mean a wave equation

$$(1.19) \quad \partial_{tt}u - \Delta u + \sum_{j=1}^m V_j(x - \vec{v}_j t)u = 0,$$

$$u|_{t=0} = g, \quad \partial_t u|_{t=0} = f, \quad x \in \mathbb{R}^3,$$

where  $\vec{v}_j$ 's are distinct vectors in  $\mathbb{R}^3$  with

$$(1.20) \quad |\vec{v}_i| < 1, \quad 1 \leq i \leq m.$$

and the real potentials  $V_j$  are such that  $\forall 1 \leq j \leq m$

- 1)  $V_j$  is time-independent and decays with rate  $\langle x \rangle^{-\alpha}$  with  $\alpha > 3$
- 2) 0 is neither an eigenvalue nor a resonance of the operators

$$(1.21) \quad H_j = -\Delta + V_j(S(\vec{v}_j)x),$$

where  $S(\vec{v}_j)x = M(\vec{v}_j)B^{-1}(\vec{v}_j)(x, 0)^T$ .

Recall that  $\psi$  is a resonance at 0 if it is a distributional solution of the equation  $H_k\psi = 0$  which belongs to the space  $L^2(\langle x \rangle^{-\sigma} dx) := \left\{ f : \langle x \rangle^{-\sigma} f \in L^2 \right\}$  for any  $\sigma > \frac{1}{2}$ , but not for  $\sigma = \frac{1}{2}$ .

*Remark.* The construction of  $S(\vec{v}_j)$  is clear from the change between different frames under Lorentz transformations. In our concrete problem below (1.24),  $S(\vec{v}_j)$  can be written down explicitly.

To simplify our argument, throughout this paper, we discuss the wave equation with a charge transfer Hamiltonian in the sense of Definition 1.1 with  $m = 2$ , a stationary  $V_1$  and a  $V_2$  moving along  $\vec{e}_1$  with speed  $|v| < 1$ , i.e., the velocity is

$$(1.22) \quad \vec{v} = (v, 0, 0).$$

Under this setting, by Definition 1.1,

$$(1.23) \quad H_1 = -\Delta + V_1(x),$$

and

$$(1.24) \quad H_2 = -\Delta + V_2\left(\sqrt{1 - |v|^2}x_1, x_2, x_3\right).$$

It is easy to see that our arguments work for  $m > 2$ .

An indispensable tool we need to study the charge transfer model is the Lorentz transformation. Throughout this paper, we apply Lorentz transformations  $L$  with respect to a moving frame with speed  $|v| < 1$  along the  $x_1$  direction. After we apply the Lorentz transformation  $L$ , under the new coordinates,  $V_2$  is stationary meanwhile  $V_1$  will be moving.

Writing down the Lorentz transformation  $L$  explicitly, we have

$$(1.25) \quad \begin{cases} t' = \gamma(t - vx_1) \\ x'_1 = \gamma(x_1 - vt) \\ x'_2 = x_2 \\ x'_3 = x_3 \end{cases}$$

with

$$(1.26) \quad \gamma = \frac{1}{\sqrt{1 - |v|^2}}.$$

We can also write down the inverse transformation of the above:

$$(1.27) \quad \begin{cases} t = \gamma(t' + vx'_1) \\ x_1 = \gamma(x'_1 + vt') \\ x_2 = x'_2 \\ x_3 = x'_3 \end{cases}.$$

Under the Lorentz transformation  $L$ , if we use the subscript  $L$  to denote a function with respect to the new coordinate  $(x', t')$ , we have

$$(1.28) \quad u_L(x'_1, x'_2, x'_3, t') = u(\gamma(x'_1 + vt'), x'_2, x'_3, \gamma(t' + vx'_1))$$

and

$$(1.29) \quad u(x, t) = u_L(\gamma(x_1 - vt), x_2, x_3, \gamma(t - vx_1)).$$

Let  $w_1, \dots, w_m$  and  $m_1, \dots, m_\ell$  be the normalized bound states of  $H_1$  and  $H_2$  associated with the negative eigenvalues  $-\lambda_1^2, \dots, -\lambda_m^2$  and  $-\mu_1^2, \dots, -\mu_\ell^2$  respectively (notice that by our assumptions, 0 is not an eigenvalue). In other words, we assume

$$(1.30) \quad H_1 w_i = -\lambda_i^2 w_i, \quad w_i \in L^2, \lambda_i > 0.$$

$$(1.31) \quad H_2 m_i = -\mu_i^2 m_i, \quad m_i \in L^2, \mu_i > 0.$$

We denote by  $P_b(H_1)$  and  $P_b(H_2)$  the projections on the the bound states of  $H_1$  and  $H_2$ , respectively, and let  $P_c(H_i) = Id - P_b(H_i)$ ,  $i = 1, 2$ . To be more explicit, we have

$$(1.32) \quad P_b(H_1) = \sum_{i=1}^m \langle \cdot, w_i \rangle w_i, \quad P_b(H_2) = \sum_{j=1}^{\ell} \langle \cdot, m_j \rangle m_j.$$

In order to study the equation with time-dependent potentials, we need to introduce a suitable projection. Again, with Lorentz transformations  $L$  associated with the moving potential  $V_2(x - vt)$ , we use the subscript  $L$  to denote a function under the new frame  $(x', t')$ .

**Definition 1.2** (Scattering states). Let

$$(1.33) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x - vt)u = 0,$$

with initial data

$$(1.34) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

If  $u$  also satisfies

$$(1.35) \quad \|P_b(H_1)u(t)\|_{L_x^2} \rightarrow 0, \quad \|P_b(H_2)u_L(t')\|_{L_{x'}^2} \rightarrow 0 \quad t, t' \rightarrow \infty,$$

we call it a scattering state.

*Remark 1.3.* Clearly, the set of  $(g, f) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$  which produces a scattering state forms a subspace of  $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ . We will see a detailed discussion on this subspace later on in Section 6.

*Remark 1.4.* Notice that in order to perform Lorentz transformations, one needs the existence of global solutions. The existence and the uniqueness of global solutions to wave equations with more general time-dependent potentials are presented by contraction arguments in [GC2].

With the above preparations, we state our main results. First of all, we have Strichartz estimates:

**Theorem 1.5** (Strichartz estimates). *Suppose  $u$  is a scattering state in the sense of Definition 1.2 and solves*

$$(1.36) \quad \partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

*with initial data*

$$(1.37) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then for  $p > 2$  and  $(p, q)$  satisfying*

$$(1.38) \quad \frac{1}{2} = \frac{1}{p} + \frac{3}{q},$$

*we have*

$$(1.39) \quad \|u\|_{L_t^p([0, \infty), L_x^q)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

The above theorem is extended to the inhomogeneous case in Section 6.

Secondly, one has the energy estimate:

**Theorem 1.6** (Energy estimate). *Suppose  $u$  is a scattering state in the sense of Definition 1.2 and solves*

$$(1.40) \quad \partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

*with initial data*

$$(1.41) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then we have*

$$(1.42) \quad \sup_{t \geq 0} (\|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}) \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

Associated with the energy estimate, we also have the local energy decay:

**Theorem 1.7.** (*Local energy decay*) *Suppose  $u$  is a scattering state in the sense of Definition 1.2 and solves*

$$(1.43) \quad \partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

*with initial data*

$$(1.44) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then for  $\forall \epsilon > 0$ ,  $|\mu| < 1$ , we have*

$$\left\| (1 + |x - \mu t|)^{-\frac{1}{2} - \epsilon} (|\nabla u| + |u_t|) \right\|_{L_{t,x}^2} \lesssim_{\mu, \epsilon} \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

Even more importantly, we obtain the endpoint reversed Strichartz estimates for  $u$ .

**Theorem 1.8** (Endpoint Strichartz estimates). *Suppose  $u$  is a scattering state in the sense of Definition 1.2 and solves*

$$(1.45) \quad \partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

*with initial data*

$$(1.46) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then*

$$(1.47) \quad \sup_{x \in \mathbb{R}^3} \int_0^\infty |u(x, t)|^2 dt \lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2,$$

and

$$(1.48) \quad \sup_{x \in \mathbb{R}^3} \int_0^\infty |u(x+vt, t)|^2 dt \lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

With the endpoint estimate along  $(x+vt, t)$ , one can derive the boundedness of the total energy. We denote the total energy of the system as

$$(1.49) \quad E(t) = \int |\nabla_x u|^2 + |\partial_t u|^2 + V_1 |u|^2 + V_2(x-vt) |u|^2 dx.$$

**Corollary 1.9** (Boundedness of the total energy). *Suppose  $u$  is a scattering state in the sense of Definition 1.2 and solves*

$$(1.50) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x-vt)u = 0$$

with initial data

$$(1.51) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

Assume

$$(1.52) \quad \|\nabla V_2\|_{L^1} < \infty,$$

then  $E(t)$  is bounded by the initial energy independently of  $t$ ,

$$(1.53) \quad \sup_{t \geq 0} E(t) \lesssim \|(g, f)\|_{\dot{H}^1 \times L^2}^2.$$

**1.3. Main ideas.** Here we briefly discuss the main ideas in our analysis and sketch our proofs. We follow the philosophy from Rodnianski-Schlag [RS] that *local decay estimates* imply Strichartz estimates. The main stream of ideas is that *the endpoint Strichartz estimate* implies weighted estimates, based on which we can derive Strichartz estimates, energy estimates, the local energy decay and the boundedness of the total energy.

An essential step to approach wave equations with moving potentials is to understand the change of energy under Lorentz transformations. In subsection 2.2, we show that the energy along a space-like slanted line stays comparable to the energy of the initial data. This in particular implies that under Lorentz transformations, the initial energy with respect to the new frame is comparable to the initial energy of the original frame. As a byproduct, we can also obtain Agmon's estimates for the decay of eigenfunctions. The arguments hold for all dimensions and even for wave equations with time-dependent potentials, cf. [GC2].

In order to handle time-dependent potentials, we need a time-dependent weight in the local decay estimate, see Chen [GC1]. More precisely, we will show that for  $|v| < 1$ ,

$$(1.54) \quad \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^\alpha} u^2(x, t) dx dt \lesssim \|(g, f)\|_{\dot{H}^1 \times L^2}^2$$

and

$$(1.55) \quad \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{1}{\langle x-vt \rangle^\alpha} u^2(x, t) dx dt \lesssim \|(g, f)\|_{\dot{H}^1 \times L^2}^2.$$

We notice that for  $\alpha > 3$ ,

$$(1.56) \quad \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^\alpha} u^2(x, t) dx dt \lesssim \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}} u^2(x, t) dt$$

and

$$(1.57) \quad \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{1}{\langle x-vt \rangle^\alpha} u^2(x, t) dx dt \lesssim \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}} u^2(x+vt, t) dt$$

which are in the form of *endpoint reversed Strichartz estimates*. But we also need to *integrate over a time-like slanted line*. These are carefully analyzed in Section 3. Intuitively, the reversed Strichartz is based on the fact that the fundamental solutions of the wave equation in  $\mathbb{R}^3$  is supported on the light cone. For fixed  $x$ , the propagation will only meet the light cone once. Meanwhile, away from the light cone, the solution

decays fast. We note that a time-like slanted line will also only intersect the light cone only once, hence we should have the same endpoint estimate along it. Our analysis crucially relies on these types of estimates. Many estimates in Section 3 also hold for more general trajectories provided that their speeds are strictly less than 1.

After performing the Lorentz transformation  $L$ , we have

$$(1.58) \quad u_L(x'_1, x'_2, x'_3, t') = u(\gamma(x'_1 + vt'), x'_2, x'_3, \gamma(t' + vx'_1))$$

and

$$(1.59) \quad u(x, t) = u_L(\gamma(x_1 - vt), x_2, x_3, \gamma(t - vx_1)).$$

It is crucial to notice that from the expressions above, the standard endpoint Strichartz estimate for  $u$  is equivalent to the endpoint Strichartz estimate along a slanted line for  $u_L$  and vice versa. It is important to note that with the above fact, we can always apply Lorentz transformations to exchange different frames if we consider several endpoint Strichartz estimates together.

Based on the observations above, we apply a bootstrap procedure for the case with two potentials. Let

$$u^S(x, t) = u(x + vt, t).$$

For a scattering state in the sense of Definition 1.2, we show that the bootstrap assumptions with big constants  $C_1(T)$  and  $C_2(T)$ ,

$$(1.60) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u(x, t)|^2 dt \leq C_1(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

and

$$(1.61) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u^S(x, t)|^2 dt \leq C_2(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2,$$

imply

$$(1.62) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u(x, t)|^2 dt \leq \left( \tilde{C}_1 + \frac{1}{2} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

and

$$(1.63) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u^S(x, t)|^2 dt \leq \left( \tilde{C}_2 + \frac{1}{2} C_2(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

Then we can conclude

$$(1.64) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u(x, t)|^2 dt \leq C_1 (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

and

$$(1.65) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u^S(x, t)|^2 dt \leq C_2 (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2,$$

for some constants  $C_1$  and  $C_2$  independent of  $T$  by the bootstrap argument. Therefore, as we pointed out above, we obtain two local decay estimates (1.54) and (1.55). To run the bootstrap argument, we use the fact that the distance between  $V_1$  and  $V_2$  becomes larger and larger and both potentials are of short-range. Therefore, for different regions in  $\mathbb{R}^3$ , the evolution will be dominated by different Hamiltonians. To make this intuition precise, in Section 4, we apply a partition of unity to carry out the decomposition into channels. For each channel, we use Duhamel's formula to compare the evolution to the associated dominating Hamiltonian. For every dominating Hamiltonian, both of the endpoint estimates hold. In each Duhamel expansion, based on the fact that  $V_1$  and  $V_2$  move far away from each other, it suffices to consider the endpoint estimates of the following integrals,



$$(1.66) \quad k_A(x, t) := \int_0^{t-A} \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_c F \, ds.$$

and

$$k_A^S(x, t) := k_A(x + vt, t).$$

From Section 3, we have

$$(1.67) \quad \begin{aligned} \|k_A\|_{L_x^\infty L_t^2[A, T]} &= \left\| \int_0^{t-A} \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_c F \, ds \right\|_{L_x^\infty L_t^2[A, T]} \\ &\lesssim \frac{1}{A} \left( \|F\|_{L_x^1 L_t^2} + \|F\|_{L_x^{\frac{3}{2}, 1} L_t^2} \right) \end{aligned}$$

and

$$(1.68) \quad \|k_A^S(x, t)\|_{L_x^\infty L_t^2[A, T]} \lesssim \frac{1}{A} \left( \|F\|_{L_x^{\frac{3}{2}, 1} L_t^2} + \|F\|_{L_x^1 L_t^2} \right).$$

Therefore for  $A > 0$  large but independent of  $T$ , this term can be absorbed to the left-hand side to improve our bootstrap assumptions.

From (1.54) and (1.55), Strichartz estimates follows from the general scheme introduced in Rodnianski-Schlag [RS, LSch].

**Notation.** “ $A := B$ ” or “ $B =: A$ ” is the definition of  $A$  by means of the expression  $B$ . We use the notation  $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ . The bracket  $\langle \cdot, \cdot \rangle$  denotes the distributional pairing and the scalar product in the spaces  $L^2, L^2 \times L^2$ . For positive quantities  $a$  and  $b$ , we write  $a \lesssim b$  for  $a \leq Cb$  where  $C$  is some prescribed constant. Also  $a \simeq b$  for  $a \lesssim b$  and  $b \lesssim a$ . We denote  $B_R(x)$  the open ball of centered at  $x$  with radius  $R$  in  $\mathbb{R}^3$ . We also denote by  $\chi$  a standard  $C^\infty$  cut-off function, that is  $\chi(x) = 1$  for  $|x| \leq 1$ ,  $\chi(x) = 0$  for  $|x| > 2$  and  $0 \leq \chi(x) \leq 1$  for  $1 \leq |x| \leq 2$ .

**Organization.** The paper is organized as follows: In Section 2, we discuss some preliminary results for the free wave equation and the wave equation with a stationary potential. We will also analyze the change of the energy under Lorentz transformations. In Section 3, estimates of homogeneous and inhomogeneous forms of wave equations along time-like slanted lines will be discussed. In Section 4 and Section 5, we show Strichartz estimates, energy estimates, the local energy decay and the boundedness of the total energy for a scattering state to the wave equation with a charge transfer Hamiltonian. In order to consider nonlinear applications, in Section 6 we discuss inhomogeneous Strichartz estimates. Finally, in Section 7, we confirm that a scattering state indeed scatters to a solution to the free wave equation.

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## 2. PRELIMINARIES

In this section, we present some preliminary results on wave equations to prepare further discussions in later sections. Throughout, we will only consider equations in  $\mathbb{R}^3$ .

**2.1. Strichartz estimates and local energy decay.** We start with the regular Strichartz estimates for free wave equations.

Consider the free wave equation,

$$(2.1) \quad \partial_{tt}u - \Delta u = F$$

with initial data

$$(2.2) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

We can write down the solution using the Fourier transform:

$$(2.3) \quad u = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}f + \cos(t\sqrt{-\Delta})g + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(s)ds.$$

It obeys the energy inequality,

$$(2.4) \quad E_F(t) = \int_{\mathbb{R}^3} |\partial_t u(t)|^2 + |\nabla u(t)|^2 dx \lesssim \int_{\mathbb{R}^3} |f|^2 + |\nabla g|^2 dx + \int_0^t \int_{\mathbb{R}^3} |F(s)|^2 dx ds.$$

We also have the well-known dispersive estimates for the free wave equation ( $H_0 = -\Delta$ ) on  $\mathbb{R}^3$ :

$$(2.5) \quad \left\| \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}f \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{|t|} \|\nabla f\|_{L^1(\mathbb{R}^3)},$$

$$(2.6) \quad \left\| \cos(t\sqrt{-\Delta})g \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{|t|} \|\Delta g\|_{L^1(\mathbb{R}^3)}.$$

Notice that the estimate (2.6) is slightly different from the estimates commonly in the literature. For example, in Krieger-Schlag [KS], one needs the  $L^1$  norm of  $D^2g$  instead of  $\Delta g$ . One can find a detailed proof in [GC2] based on an idea similar to the endpoint reversed Strichartz estimate.

Strichartz estimates can be derived abstractly from these dispersive inequalities and the energy inequality. The following theorem is standard. One can find a detailed proof in, for example, Keel-Tao [KT].

**Theorem 2.1** (Strichartz estimates). *Suppose*

$$(2.7) \quad \partial_{tt}u - \Delta u = F$$

with initial data

$$(2.8) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

Then for  $p, a > \frac{2}{s}$ ,  $(p, q)$ ,  $(a, b)$  satisfying

$$(2.9) \quad \frac{3}{2} - s = \frac{1}{p} + \frac{3}{q}$$

$$(2.10) \quad \frac{3}{2} - s = \frac{1}{a} + \frac{3}{b}$$

we have

$$(2.11) \quad \|u\|_{L_t^p L_x^q} \lesssim \|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s-1}} + \|F\|_{L_t^{a'} L_x^{b'}}$$

where  $\frac{1}{a} + \frac{1}{a'} = 1$ ,  $\frac{1}{b} + \frac{1}{b'} = 1$ .

The endpoint  $(p, q) = (2, \infty)$  can be recovered for radial functions in Klainerman-Machedon [KM] for the homogeneous case and Jia-Liu-Schlag-Xu [JLSX] for the inhomogeneous case, The endpoint estimate can also be obtained when a small amount of smoothing (either in the Sobolev sense, or in relaxing the integrability) is applied to the angular variable, see Machihara-Nakamura-Nakanishi-Ozawa [MNNO].

The regular Strichartz estimates fail at the endpoint. But if one switches the order of space-time integration, it is possible to estimate the solution using the fact that the solution decays quickly away from the light

cone. Therefore, we introduce reversed Strichartz estimates. Since we will only use the endpoint reversed Strichartz estimate, we will restrict our focus to that case. The detailed proof for free equations is presented for the sake of completeness.

**Theorem 2.2** (Endpoint reversed Strichartz estimate). *Suppose*

$$(2.12) \quad \partial_{tt}u - \Delta u = F$$

*with initial data*

$$(2.13) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then*

$$(2.14) \quad \|u\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2},$$

*and for  $T > 0$ ,*

$$(2.15) \quad \|u\|_{L_x^\infty L_t^2[0,T]} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}.$$

*Proof.* Writing down  $u$  explicitly, we have

$$(2.16) \quad u = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f + \cos(t\sqrt{-\Delta}) g + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds.$$

We will analyze each term separately. By symmetry, we may assume that  $t \geq 0$ .

For the first term,

$$(2.17) \quad \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f = \frac{1}{4\pi t} \int_{|x-y|=t} f(y) \sigma(dy).$$

So in polar coordinates,

$$\begin{aligned} \left\| \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f \right\|_{L_t^2}^2 &\lesssim \int_0^\infty \left( \int_{\mathbb{S}} f(x+r\omega) r d\omega \right)^2 dr \\ &\lesssim \left( \int_0^\infty \int_{\mathbb{S}} f(x+r\omega)^2 r^2 d\omega dr \right) \left( \int_{\mathbb{S}^2} d\omega \right) \\ &\lesssim \|f\|_{L^2}^2. \end{aligned}$$

Therefore,

$$(2.18) \quad \left\| \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2}.$$

For the second term,

$$\begin{aligned} \left\| \cos(t\sqrt{-\Delta}) g \right\|_{L_t^2}^2 &= \int_0^\infty \left( \int_{\mathbb{S}^2} g(x+r\omega) d\omega + r \partial_r g(x+r\omega) d\omega \right)^2 dr \\ &\lesssim \int_0^\infty \int_{\mathbb{S}^2} (g(x+r\omega) d\omega)^2 dr + \int_0^\infty \int_{\mathbb{S}^2} (\partial_r g(x+r\omega) d\omega)^2 r^2 dr \\ &\lesssim \left( \int_0^\infty \int_{\mathbb{S}^2} g(x+r\omega)^2 d\omega dr \right) \left( \int_{\mathbb{S}^2} d\omega \right) \\ &\quad + \left( \int_0^\infty \int_{\mathbb{S}^2} \partial_r g(x+r\omega)^2 d\omega r^2 dr \right) \left( \int_{\mathbb{S}^2} d\omega \right) \\ &\lesssim \|\nabla g\|_{L^2}^2, \end{aligned}$$

where for the last inequality, we applied Hardy's inequality

$$(2.19) \quad \left\| |x|^{-1} g \right\|_{L^2} \lesssim \|g\|_{\dot{H}^1}.$$

Therefore,

$$(2.20) \quad \left\| \cos \left( t\sqrt{-\Delta} \right) g \right\|_{L_x^\infty L_t^2} \lesssim \|\nabla g\|_{L^2}.$$

For the third term,

$$\begin{aligned} \left\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds \right\|_{L_t^2} &= \left\| \int_0^t \int_{|x-y|=t-s} \frac{1}{|x-y|} F(y, s) \sigma(dy) ds \right\|_{L_t^2} \\ &= \left\| \int_{|x-y|\leq t} \frac{1}{|x-y|} F(y, t-|x-y|) dy \right\|_{L_t^2} \\ &\lesssim \int \frac{1}{|x-y|} \|F(y, t-|x-y|)\|_{L_t^2} dy \\ &\lesssim \sup_{x \in \mathbb{R}^3} \int \frac{1}{|x-y|} \|F(y, t)\|_{L_t^2} dy \\ &\lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2}, \end{aligned}$$

where we applied Minkowski's inequality in the third line. Here  $L^{\frac{3}{2},1}$  is the Lorentz space. In the last inequality, we applied the following fact:

$$(2.21) \quad \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|h(x)|}{|x-y|} dx = \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|h(x-y)|}{|x|} dx.$$

Then for fixed  $y$ , we apply Hölder's inequality for Lorentz spaces, see Theorem 3.5 in O'Neil [ON],

$$(2.22) \quad \int_{\mathbb{R}^3} \frac{|h(x-y)|}{|x|} dx = \left\| \frac{|h(x-y)|}{|x|} \right\|_{L^{1,1}} \lesssim \left\| \frac{1}{|x|} \right\|_{L^{3,\infty}} \|h(x-y)\|_{L_x^{\frac{3}{2},1}}.$$

Notice that

$$(2.23) \quad \|h(x-y)\|_{L_x^{\frac{3}{2},1}} = \|h(x)\|_{L_x^{\frac{3}{2},1}}.$$

Therefore,

$$(2.24) \quad \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|h(x)|}{|x-y|} dx \lesssim \|h\|_{L_x^{\frac{3}{2},1}}.$$

Hence

$$(2.25) \quad \left\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds \right\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2}.$$

We also notice that for  $T > 0$ ,

$$\left\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds \right\|_{L_t^2[0,T]} \lesssim \left\| \int_{|x-y|\leq t} \frac{1}{|x-y|} F(y, t-|x-y|) dy \right\|_{L_t^2[0,T]}$$

with

$$0 \leq t - |x-y| \leq T,$$

whence

$$\begin{aligned}
\left\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds \right\|_{L_t^2[0,T]} &\lesssim \left\| \int_{|x-y|\leq t} \frac{1}{|x-y|} F(y, t-|x-y|) dy \right\|_{L_t^2[0,T]} \\
&\lesssim \int \frac{1}{|x-y|} \|F(y, t)\|_{L_t^2[0,T]} dy \\
&\lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}.
\end{aligned}$$

Therefore,

$$(2.26) \quad \left\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds \right\|_{L_t^2[0,T]} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}.$$

The theorem is proved.  $\square$

The above results from Theorem 2.1 and Theorem 2.2 can be generalized to wave equations with real stationary potentials.

Denote

$$(2.27) \quad H = -\Delta + V,$$

where the potential  $V$  satisfies the assumption in Definition 1.1.

Consider the wave equation with potential in  $\mathbb{R}^3$ :

$$(2.28) \quad \partial_{tt}u - \Delta u + Vu = 0$$

with initial data

$$(2.29) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

One can write down the solution to it explicitly:

$$(2.30) \quad u = \frac{\sin(t\sqrt{H})}{\sqrt{H}} f + \cos(t\sqrt{H}) g.$$

Let  $P_b$  be the projection onto the point spectrum of  $H$ ,  $P_c = I - P_b$  be the projection onto the continuous spectrum of  $H$ .

With the above setting, we formulate the results from [BecGo].

**Theorem 2.3** (Strichartz and reversed Strichartz estimates). *Suppose  $H$  has neither eigenvalues nor resonances at zero. Then for all  $0 \leq s \leq 1$ ,  $p > \frac{2}{s}$ , and  $(p, q)$  satisfying*

$$(2.31) \quad \frac{3}{2} - s = \frac{1}{p} + \frac{3}{q}$$

*we have*

$$(2.32) \quad \left\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f + \cos(t\sqrt{H}) P_c g \right\|_{L_t^p L_x^q} \lesssim \|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s-1}}.$$

*For the endpoint reversed Strichartz estimate, we have*

$$(2.33) \quad \left\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f + \cos(t\sqrt{H}) P_c g \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1},$$

$$(2.34) \quad \left\| \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_c F(s) ds \right\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2},$$

and for  $T > 0$ ,

$$(2.35) \quad \left\| \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_c F(s) ds \right\|_{L_x^\infty L_t^2[0,T]} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}.$$

One can find detailed arguments and more estimates in [BecGo]. The above theorem can also be established by passing the estimates for free wave equations in Theorem 2.1 and Theorem 2.2 to the perturbed case via the structure of wave operators. This general strategy is discussed in detail in [GC2].

*Remark 2.4.* In [BecGo], the above theorem is shown for potentials  $V$  with a finite global Kato norm. The Kato space  $K$  is the Banach space of measures with the property that

$$(2.36) \quad \|V\|_K = \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(x)|}{|x-y|} dx.$$

They consider the space of potentials  $V$  which are taken in the Kato norm closure of the set of bounded, compactly supported functions, which is denoted by  $K_0$ . Note that from estimate (2.24),  $L_x^{\frac{3}{2},1} \subset K$ .

Next, we formulate one fundamental mechanism of wave equations: local energy decay. It suffices to consider the half-wave operator.

**Theorem 2.5** (Local energy decay).  *$\forall \epsilon > 0$ , one has*

$$(2.37) \quad \left\| (1+|x|)^{-\frac{1}{2}-\epsilon} e^{it\sqrt{-\Delta}} f \right\|_{L_{t,x}^2} \lesssim_\epsilon \|f\|_{L_x^2}.$$

See Corollary 2.10 for a more general formulation with time-dependent weight.

The following Christ-Kiselev Lemma is important in our derivation of Strichartz estimates.

**Lemma 2.6** (Christ-Kiselev). *Let  $X, Y$  be two Banach spaces and let  $T$  be a bounded linear operator from  $L^\beta(\mathbb{R}^+; X)$  to  $L^\gamma(\mathbb{R}^+; Y)$ , such that*

$$(2.38) \quad Tf(t) = \int_0^\infty K(t,s)f(s) ds.$$

*Then the operator*

$$(2.39) \quad \tilde{T} = \int_0^t K(t,s)f(s) ds$$

*is bounded from  $L^\beta(\mathbb{R}^+; X)$  to  $L^\gamma(\mathbb{R}^+; Y)$  provided  $\beta < \gamma$ , and the*

$$(2.40) \quad \|\tilde{T}\| \leq C(\beta, \gamma) \|T\|$$

*with*

$$(2.41) \quad C(\beta, \gamma) = \left(1 - 2^{\frac{1}{\gamma} - \frac{1}{\beta}}\right)^{-1}.$$

**2.2. Lorentz Transformations and Energy.** In this paper, Lorentz transformations will be important for us to reduce some estimates to stationary cases. In order to approach our problem from the viewpoint of Lorentz transformations, the first natural step is to understand the change of energy under Lorentz transformations. In this subsection, we show that under Lorentz transformations, the energy stays comparable to that of the initial data. Recall that after we apply the Lorentz transformation, for function  $u$ , under the new coordinates, we denote

$$(2.42) \quad u_L(x'_1, x'_2, x'_3, t') = u(\gamma(x'_1 + vt'), x'_2, x'_3, \gamma(t' + vx'_1)).$$

Now let  $u$  to be a solution to some wave equation and set  $t' = 0$ . We notice that in order to show under Lorentz transformations, the energy stays comparable to that of the initial data up to an absolute constant, it suffices to prove

$$(2.43) \quad \begin{aligned} & \int |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 dx \\ & \simeq \int |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 dx. \end{aligned}$$

provided  $|v| < 1$ .

Throughout this subsection, we will assume all functions are smooth and decay fast. We will obtain estimates independent of the additional smoothness assumption. It is easy to pass the estimates to general cases with a density argument.

*Remark 2.7.* One can observe that all discussions in this section hold for  $\mathbb{R}^n$ . We choose  $n = 3$  since we will only consider the charge transfer model in  $\mathbb{R}^3$  in later parts of this paper.

In this subsection, a more general situation is analyzed. We consider wave equations with time-dependent potentials

$$(2.44) \quad \partial_{tt}u - \Delta u + V(x, t)u = 0$$

under some uniform decay conditions

$$(2.45) \quad |V(x, \mu x_1)| \lesssim \frac{1}{\langle x \rangle^3}$$

uniformly for  $0 \leq |\mu| \leq 1$ . These in particular apply to wave equations with moving potentials with speed strictly less than 1. For example,

$$(2.46) \quad V(x, t) = V(x - vt)$$

with

$$(2.47) \quad |V(x)| \lesssim \frac{1}{\langle x \rangle^3}.$$

**Theorem 2.8.** *Let  $|v| < 1$ . Suppose*

$$(2.48) \quad \partial_{tt}u - \Delta u + V(x, t)u = 0$$

and

$$(2.49) \quad |V(x, \mu x_1)| \lesssim \frac{1}{\langle x \rangle^3}$$

uniformly with respect to  $0 \leq |\mu| < 1$ . Then

$$(2.50) \quad \begin{aligned} & \int |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 dx \\ & \simeq \int |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 dx, \end{aligned}$$

where the implicit constant depends on  $v$  and  $V$ .

*Proof.* Up to performing a Lorentz transformation or a change of variable, it suffices to show

$$(2.51) \quad \begin{aligned} & \int |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 dx \\ & \lesssim \int |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 dx. \end{aligned}$$

Set

$$(2.52) \quad E_1(\mu) = \int |\nabla_x u(x_1, x_2, x_3, \mu x_1)|^2 dx,$$

$$(2.53) \quad E_2(\mu) = \int |\partial_t u(x_1, x_2, x_3, \mu x_1)|^2 dx.$$

In the following computations, for a function  $f(x, t)$ , we use the short-hand notation

$$\int f dx = \int_{\mathbb{R}^3} f(x_1, x_2, x_3, \mu x_1) dx.$$

Then

$$(2.54) \quad \begin{aligned} \frac{dE_1}{d\mu} &= 2 \int x_1 \nabla_x u(x_1, x_2, x_3, \mu x_1) \nabla_x u_t(x_1, x_2, x_3, \mu x_1) dx \\ &= 2 \int x_1 \nabla_x u \nabla_x u_t dx \end{aligned}$$

and

$$(2.55) \quad \begin{aligned} \frac{dE_2}{d\mu} &= 2 \int x_1 \partial_t u(x_1, x_2, x_3, \mu x_1) \partial_{tt} u(x_1, x_2, x_3, \mu x_1) dx \\ &= 2 \int x_1 \partial_t u \partial_{tt} u dx. \end{aligned}$$

Integration by parts in (2.54) gives

$$(2.56) \quad \frac{dE_1}{d\mu} = -2 \int \partial_{x_1} u \cdot u_t dx - 2 \int x_1 \Delta u \cdot u_t dx - 2\mu \int x_1 \partial_{x_1} u_t \cdot u_t dx.$$

And using the fact  $u$  solve the wave equation implies

$$(2.57) \quad \frac{dE_2}{d\mu} = 2 \int x_1 \partial_t u \cdot \Delta u dx - 2 \int x_1 \partial_t u \cdot V u dx.$$

Consider the following integral appearing as the last term in (2.56),

$$(2.58) \quad \int x_1 \partial_{x_1} u_t \cdot u_t dx.$$

Integration by parts in  $x$ , one has

$$\int x_1 \partial_{x_1} u_t \cdot u_t dx = - \int |u_t|^2 dx - \int x_1 \partial_{x_1} u_t \cdot u_t dx - \mu \int x_1 u_t \cdot u_{tt} dx.$$

Therefore,

$$(2.59) \quad \int x_1 \partial_{x_1} u_t \cdot u_t dx = -\frac{1}{2} \int |u_t|^2 dx - \frac{\mu}{4} \frac{dE_2}{d\mu}.$$

Combining identities (2.56), (2.57) and (2.59) together, we have

$$E_1'(\mu) + \left(1 - \frac{\mu^2}{2}\right) E_2'(\mu) = H(\mu),$$

where

$$(2.60) \quad H(\mu) = -2 \int \partial_{x_1} u \cdot u_t dx - 2 \int x_1 \partial_t u \cdot V u dx + \mu \int |u_t|^2 dx.$$

By Cauchy-Schwarz and Hardy's inequality,

$$|H(\mu)| \lesssim E_1(\mu) + E_2(\mu),$$

and hence

$$(2.61) \quad \left| E_1'(\mu) + \left(1 - \frac{\mu^2}{2}\right) E_2'(\mu) \right| \lesssim E_1(\mu) + E_2(\mu).$$



Setting

$$E_3(\mu) = E_1(\mu) + \left(1 - \frac{\mu^2}{2}\right) E_2(\mu),$$

one has

$$E_3'(\mu) = E_1'(\mu) + \left(1 - \frac{\mu^2}{2}\right) E_2'(\mu) - \mu E_2(\mu)$$

and

$$\left|E_3'(\mu)\right| \lesssim E_1(\mu) + E_2(\mu) + \mu E_2(\mu) \lesssim E_1(\mu) + E_2(\mu)$$

by (2.61).

Since  $0 \leq \mu < 1$ ,

$$(2.62) \quad E_1(\mu) + E_2(\mu) \lesssim E_1(\mu) + \left(1 - \frac{\mu^2}{2}\right) E_2(\mu) = E_3(\mu),$$

so

$$(2.63) \quad \left|E_3'(\mu)\right| \lesssim E_3(\mu).$$

Applying Grönwall's inequality,

$$(2.64) \quad E_1(\mu) + E_2(\mu) \lesssim E_3(\mu) \lesssim e^\mu E_3(0) \lesssim E_1(0) + E_2(0).$$

Therefore, by the definitions of  $E_1(\mu)$  and  $E(\mu)$ , we have

$$\begin{aligned} & \int |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 dx \\ & \lesssim \int |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 dx. \end{aligned}$$

The theorem is proved.  $\square$

*Remark 2.9.* The above theorem can be also obtained by local energy conservation and the control of the energy flux. And this approach will only require the potential to decay with rate  $\langle x \rangle^{-2}$ . See [GC2] for more details.

Applying Theorem 2.8 in the setting of Theorem 2.5, we obtain a more general formulation of the local energy decay estimate.

**Corollary 2.10.**  $\forall \epsilon > 0$   $|\vec{\mu}| < 1$ , one has

$$(2.65) \quad \left\| (1 + |x - \vec{\mu}t|)^{-\frac{1}{2}-\epsilon} e^{it\sqrt{-\Delta}} f \right\|_{L_{t,x}^2} \lesssim_\epsilon \|f\|_{L_x^2}.$$

As a by product of Theorem 2.8, we obtain Agmon's estimates [Agmon] for the decay of eigenfunctions associated with negative eigenvalues. One can find a detailed proof in [GC2].

### 3. ESTIMATES ALONG SLANTED LINES

In order to obtain reversed Strichartz estimates for wave equations with moving potentials, we need to understand the analogous estimates along slanted lines. With the results from subsection 2.2, we first consider the estimates along slanted lines for free wave equations. For the free evolution, the results can be obtained by explicit calculations with the Kirchhoff formula or the Fourier transforms, for example see the calculations in [GC2]. In this section, we will approach those estimates with a viewpoint of Lorentz transformations. The reason is that this approach will be more consistent with our construction later on.

**3.1. Free wave equations.** First of all, we will consider

$$(3.1) \quad \partial_{tt}u - \Delta u = 0,$$

with initial data

$$(3.2) \quad u(x, 0) = g, \quad u_t(x, 0) = f(x).$$

We can write

$$(3.3) \quad u(x, t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f + \cos(t\sqrt{-\Delta}) g.$$

By our preliminary discussions in Theorem 2.2, we know

$$(3.4) \quad \|u\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L_x^2} + \|\nabla g\|_{L_x^2}.$$

We consider an analogous estimate to (3.4) along slanted lines. To be more concrete, we integrate  $u^2$  along slanted lines

$$(3.5) \quad (x + vt, t) = (x_1 + vt, x_2, x_3, t)$$

Denote

$$(3.6) \quad u^S(x, t) := u(x + vt, t),$$

we estimate

$$(3.7) \quad \|u^S\|_{L_x^\infty L_t^2}.$$

**Lemma 3.1.** *Let  $|v| < 1$  and suppose  $u$  solves*

$$(3.8) \quad \partial_{tt}u - \Delta u = 0$$

*with initial data*

$$(3.9) \quad u(0) = g, \quad u_t(0) = f.$$

*Then*

$$(3.10) \quad \|u^S\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

*Proof.* Recall that performing the Lorentz transformation with respect to  $v$ , in the new frame, one has

$$(3.11) \quad u_L(x'_1, x'_2, x'_3, t') := u(\gamma(x'_1 + vt'), x'_2, x'_3, \gamma(t' + vx'_1))$$

and

$$(3.12) \quad \partial_{t't'}u_L - \Delta_{x'}u_L = 0.$$

Notice that from (3.11), to estimate the  $L_x^\infty L_t^2$  norm of

$$(3.13) \quad u^S = u(x + vt, t),$$

is equivalent to integrating of  $u_L$  along  $t'$  up to a multiplication of an absolute constant only depending on  $v$  and  $\gamma$ .

Therefore, by the endpoint reversed Strichartz estimate for  $u_L$ , we have

$$(3.14) \quad \|u^S\|_{L_x^\infty L_t^2} \lesssim \|u_L\|_{L_x^\infty L_t^2} \lesssim \|\partial_t u_L(0)\|_{L^2} + \|u_L(0)\|_{\dot{H}^1} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1},$$

where in the last inequality, we applied Theorem 2.8 with  $V \equiv 0$ . □

**3.2. Wave equations with stationary potentials.** In this subsection, we consider the perturbed Hamiltonian,

$$(3.15) \quad H = -\Delta + V,$$

and the wave equation with potential,

$$(3.16) \quad \partial_{tt}u + Hu = 0$$

with initial data

$$u(x, 0) = g, \quad u_t(x, 0) = f.$$

The results in this section can always be obtained by the related estimates for the free case via the structure formula of wave operators, cf. [GC2]. But in order to make our exposition self-contained, we will prove all estimates independent of the structure formula.

For simplicity, from now on till the end of this section, we will assume  $g = 0$ . For the other case, the analysis is similar with  $L^2$  norm replaced by  $\dot{H}^1$  norm.

**Theorem 3.2.** *Let  $|v| < 1$  and set*

$$(3.17) \quad u(x, t) = \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f.$$

*Denote*

$$(3.18) \quad u^S(x, t) := u(x + vt, t)$$

*then*

$$(3.19) \quad \|u^S\|_{L_x^\infty L_t^2} \lesssim \|P_c f\|_{L^2} \lesssim \|f\|_{L^2}.$$

*Proof.* By Duhamel's formula, we write

$$(3.20) \quad \begin{aligned} u(x, t) &= \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} P_c f - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} V \frac{\sin(s\sqrt{H})}{\sqrt{H}} P_c f \, ds, \\ &=: A + B \end{aligned}$$

Now consider the estimate along slanted lines. The estimate for  $A$  is known from the free evolution, Lemma 3.1. For the second term, we use the explicit representation of the free evolution  $\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}$ .

Set

$$(3.21) \quad g(\cdot, t) = \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) \, ds$$

along slanted lines. First of all, by our preliminary results, Theorem 2.2,

$$(3.22) \quad \|g\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2}.$$

For the estimate along slanted lines, by Kirchhoff's formula, we know

$$(3.23) \quad g^S(x, t) := g(x + vt, t) = \int_0^t \int_{|x+vt-y|=t-s} \frac{F(y, s)}{|x + vt - y|} \sigma(dy) \, ds$$

and

$$\begin{aligned}
\|g^S(x, \cdot)\|_{L_t^2} &= \left\| \int_0^t \int_{|x+vt-y|=t-s} \frac{F(y, s)}{|x+vt-y|} \sigma(dy) ds \right\|_{L_t^2} \\
(3.24) \quad &= \left\| \int_{|y|\leq t} \frac{F(x+vt-y, t-|y|)}{|y|} dy \right\|_{L_t^2} \\
&\leq \left\| \int_{\mathbb{R}^3} \frac{|F(x-y, t-|y+vt|)|}{|y+vt|} dy \right\|_{L_t^2} \\
&\leq \left\| \int_{\mathbb{R}^3} \frac{|F(x-y, t-|y+vt|)|}{\sqrt{y_2^2 + y_3^2}} dy \right\|_{L_t^2},
\end{aligned}$$

where in the third line, we used a change of variable and for the last inequality and reduce the norm of  $y$  to the norm of the component of  $y$  orthogonal to the direction of the motion.

Finally,

$$(3.25) \quad \left\| \int_{\mathbb{R}^3} \frac{F(x-y, t-|y+vt|)}{\sqrt{y_2^2 + y_3^2}} dy \right\|_{L_t^2} \leq \int_{\mathbb{R}^3} \frac{\|F(x-y, t-|y+vt|)\|_{L_t^2}}{\sqrt{y_2^2 + y_3^2}} dy$$

For fixed  $y$ , if we apply a change of variable of  $t$  here, the Jacobian is bounded by  $1 - |v|$  and  $1 + |v|$ , so

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{\|F(x-y, t-|y+vt|)\|_{L_t^2}}{\sqrt{y_2^2 + y_3^2}} dy &\lesssim \int_{\mathbb{R}^3} \frac{\|F(x-y, \cdot)\|_{L_t^2}}{\sqrt{y_2^2 + y_3^2}} dy \\
(3.26) \quad &\lesssim \|F\|_{L_{x_1}^1 L_{\widehat{x}_1}^{2,1} L_t^2}
\end{aligned}$$

where  $\widehat{x}_1$  denotes the subspace orthogonal to  $x_1$  (more generally, the subspace orthogonal to the direction of the motion). Here  $L^{2,1}$  is the Lorentz norm and the last inequality follows from Hölder's inequality of Lorentz spaces. Therefore,

$$(3.27) \quad \|g^S\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_{x_1}^1 L_{\widehat{x}_1}^{2,1} L_t^2}.$$

By a similar discussion as the estimate (2.26), we also have for  $T > 0$ ,

$$(3.28) \quad \|g^S\|_{L_x^\infty L_t^2[0,T]} \lesssim \|F\|_{L_{x_1}^1 L_{\widehat{x}_1}^{2,1} L_t^2[0,T]}.$$

With estimate (3.27), we know for  $u^S(x, t) := u(x + vt, t)$ ,

$$\begin{aligned}
\|u^S\|_{L_x^\infty L_t^2} &\lesssim \|P_c f\|_{L^2} + \left\| V \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f \right\|_{L_{x_1}^1 L_{\widehat{x}_1}^{2,1} L_t^2} \\
&\lesssim \|P_c f\|_{L^2} + \|V\|_{L_{x_1}^1 L_{\widehat{x}_1}^{2,1}} \left\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f \right\|_{L_x^\infty L_t^2} \\
(3.29) \quad &\lesssim \|P_c f\|_{L^2} + \|V\|_{L_{x_1}^1 L_{\widehat{x}_1}^{2,1}} \|P_c f\|_{L_x^2} \\
&\lesssim \|f\|_{L^2}.
\end{aligned}$$

where in the third line, we used the endpoint reversed Strichartz estimate of the wave equation with potentials as Theorem 2.3.

Therefore,

$$(3.30) \quad \|u\|_{L_x^\infty L_t^2} \lesssim \|P_c f\|_{L_x^2} \lesssim \|f\|_{L_x^2}$$

$$(3.31) \quad \|u^S\|_{L_x^\infty L_t^2} \lesssim \|P_c f\|_{L_x^2} \lesssim \|f\|_{L_x^2}$$

as claimed.  $\square$

As a byproduct, we have the following inhomogeneous estimates from (3.27) and (3.28).

**Corollary 3.3.** *For  $|v| < 1$  we have*

$$(3.32) \quad \left\| \int_0^t \int_{|x+vt-y|=t-s} \frac{F(y,s)}{|x+vt-y|} dy ds \right\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_{x_1}^1 L_{x_1}^{2,1} L_t^2},$$

and for  $T > 0$ ,

$$(3.33) \quad \left\| \int_0^t \int_{|x+vt-y|=t-s} \frac{F(y,s)}{|x+vt-y|} dy ds \right\|_{L_x^\infty L_t^2[0,T]} \lesssim \|F\|_{L_{x_1}^1 L_{x_1}^{2,1} L_t^2[0,T]}.$$

From the discussion above, we can also obtain the following truncated versions of inhomogeneous estimates which are crucial in our later bootstrap arguments.

**Corollary 3.4.** *Suppose  $A > 0$  and  $|v| < 1$ , then*

$$(3.34) \quad \sup_x \left\| \int_0^{t-A} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F ds \right\|_{L_t^2[A,\infty)} \lesssim \frac{1}{A} \|F\|_{L_x^1 L_t^2}.$$

and for  $T > 0$ ,

$$(3.35) \quad \sup_x \left\| \int_0^{t-A} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F ds \right\|_{L_t^2[A,T]} \lesssim \frac{1}{A} \|F\|_{L_x^1 L_t^2[0,T]}.$$

Similarly,

$$(3.36) \quad \sup_x \left\| \int_0^{t-A} \int_{|x+vt-y|=t-s} \frac{F(y,s)}{|x+vt-y|} \sigma(dy) ds \right\|_{L_t^2[A,\infty)} \lesssim \frac{1}{A} \|F\|_{L_x^1 L_t^2},$$

and for  $T > 0$

$$(3.37) \quad \sup_x \left\| \int_0^{t-A} \int_{|x+vt-y|=t-s} \frac{F(y,s)}{|x+vt-y|} \sigma(dy) ds \right\|_{L_t^2[A,T]} \lesssim \frac{1}{A} \|F\|_{L_x^1 L_t^2[0,T]}.$$

*Proof.* By a similar discussion above with Kirchhoff's formula,

$$(3.38) \quad \begin{aligned} \left\| \int_0^{t-A} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds \right\|_{L_t^2[A,\infty)} &= \left\| \int_{A \leq |y| \leq t} \frac{F(x-y, t-|y|)}{|y|} dy \right\|_{L_t^2[A,\infty)} \\ &\lesssim \int_{A \leq |y|} \frac{\|F(x-y, t-|y|)\|_{L_t^2}}{|y|} dy \\ &\lesssim \frac{1}{A} \|F\|_{L_x^1 L_t^2}. \end{aligned}$$

Therefore,

$$(3.39) \quad \left\| \int_0^{t-A} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds \right\|_{L_x^\infty L_t^2[A,\infty)} \lesssim \frac{1}{A} \|F\|_{L_x^1 L_t^2}.$$

With the same argument as (3.28), we also have

$$(3.40) \quad \left\| \int_0^{t-A} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds \right\|_{L_t^2[A, T]} \lesssim \frac{1}{A} \|F\|_{L_x^1 L_t^2[0, T]}.$$

Similarly to the way we derive estimates (3.32) and (3.33), one obtains

$$(3.41) \quad \left\| \int_0^{t-A} \int_{|x+vt-y|=t-s} \frac{F(y, s)}{|x+vt-y|} \sigma(dy) ds \right\|_{L_x^\infty L_t^2[A, \infty)} \lesssim \frac{1}{A} \|F\|_{L_x^1 L_t^2},$$

$$(3.42) \quad \left\| \int_0^{t-A} \int_{|x+vt-y|=t-s} \frac{F(y, s)}{|x+vt-y|} \sigma(dy) ds \right\|_{L_x^\infty L_t^2[A, T]} \lesssim \frac{1}{A} \|F\|_{L_x^1 L_t^2[0, T]}.$$

We are done.  $\square$

Next, we consider estimates in inhomogeneous forms for the perturbed evolution along slanted lines. In the following proofs, essentially, we pass the effects caused by the integration along slanted lines to the free evolution by Duhamel expansion and use the standard case for the perturbed evolution.

Define

$$(3.43) \quad k(\cdot, t) := \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_c F(s) ds.$$

Then from the endpoint reversed Strichartz estimate, Theorem 2.3, we have

$$(3.44) \quad \|k\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_x^{\frac{3}{2}, 1} L_t^2}.$$

**Theorem 3.5.** *Let  $|v| < 1$  and suppose  $H = -\Delta + V$  has neither resonances nor eigenfunctions at 0. Define*

$$(3.45) \quad k^S(x, t) = k(x + vt, t).$$

*Then we have*

$$(3.46) \quad \|k^S\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_{x_1}^1 L_{x_1'}^{2, 1} L_t^2} + \|F\|_{L_x^{\frac{3}{2}, 1} L_t^2},$$

*and for  $T > 0$ ,*

$$(3.47) \quad \|k^S\|_{L_x^\infty L_t^2[0, T]} \lesssim \|F\|_{L_{x_1}^1 L_{x_1'}^{2, 1} L_t^2[0, T]} + \|F\|_{L_x^{\frac{3}{2}, 1} L_t^2[0, T]},$$

*where  $\widehat{x_1}$  is the subspace orthogonal to  $\vec{e}_1$ .*

*Proof.* By Duhamel's formula, we write

$$(3.48) \quad \begin{aligned} \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_c F(s) &= \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_c F(s) \\ &\quad - \int_s^t \frac{\sin((t-m)\sqrt{-\Delta})}{\sqrt{-\Delta}} V \frac{\sin((m-s)\sqrt{H})}{\sqrt{H}} P_c F(s) dm. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_c F ds &= \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds \\ &\quad - \int_0^t \int_s^t \frac{\sin((t-m)\sqrt{-\Delta})}{\sqrt{-\Delta}} V \frac{\sin((m-s)\sqrt{H})}{\sqrt{H}} P_c F(s) dm ds. \end{aligned}$$

Denote

$$(3.49) \quad R(x, t) := \int_0^t \int_s^t \frac{\sin((t-m)\sqrt{-\Delta})}{\sqrt{-\Delta}} V \frac{\sin((m-s)\sqrt{H})}{\sqrt{H}} P_c F(s) dm ds$$

and

$$(3.50) \quad R^S(x, t) := R(x + vt, t).$$

Then

$$(3.51) \quad \|k^S\|_{L_x^\infty L_t^2} \lesssim \|g^S\|_{L_x^\infty L_t^2} + \|R^S\|_{L_x^\infty L_t^2},$$

where

$$(3.52) \quad g^S(x, t) = g(x + vt, t) = \int_0^t \int_{|x+vt-y|=t-s} \frac{F(y, s)}{|x+vt-y|} dy ds.$$

From Corollary 3.3, we know

$$(3.53) \quad \|g^S\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_{x_1}^1 L_{\bar{x}_1}^{2,1} L_t^2}.$$

To estimate

$$(3.54) \quad \left\| \int_0^t \int_s^t \frac{\sin((t-k)\sqrt{-\Delta})}{\sqrt{-\Delta}} V \frac{\sin((k-s)\sqrt{H})}{\sqrt{H}} P_c F(s) dk ds \right\|_{L_t^2},$$

we notice that with an exchange of the order of integration,

$$(3.55) \quad \begin{aligned} R(x, t) &= \int_0^t \int_s^t \frac{\sin((t-k)\sqrt{-\Delta})}{\sqrt{-\Delta}} V \frac{\sin((k-s)\sqrt{H})}{\sqrt{H}} P_c F(s) dk ds \\ &= \int_0^t \frac{\sin((t-k)\sqrt{-\Delta})}{\sqrt{-\Delta}} \left( \int_0^k V \frac{\sin((k-s)\sqrt{H})}{\sqrt{H}} P_c F(s) ds \right) dk. \end{aligned}$$

Then applying our estimate for the free evolution estimate in the inhomogeneous case, Corollary 3.3, we have

$$(3.56) \quad \begin{aligned} \|R^S(x, t)\|_{L_x^\infty L_t^2} &\lesssim \left\| \int_0^t V \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_c F(s) ds \right\|_{L_{x_1}^1 L_{\bar{x}_1}^{2,1} L_t^2} \\ &\lesssim \|V\|_{L_{x_1}^1 L_{\bar{x}_1}^{2,1}} \left\| \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_c F(s) ds \right\|_{L_x^\infty L_t^2} \\ &\lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2} \end{aligned}$$

where in the third inequality, we used the endpoint reversed Strichartz estimate (3.44).

Therefore, we conclude that

$$(3.57) \quad \begin{aligned} \|k^S\|_{L_x^\infty L_t^2} &\lesssim \|g^S\|_{L_x^\infty L_t^2} + \|R^S\|_{L_x^\infty L_t^2} \\ &\lesssim \|F\|_{L_{x_1}^1 L_{\bar{x}_1}^{2,1} L_t^2} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2}. \end{aligned}$$

When we restrict to  $[0, T]$ , as above, we can obtain

$$(3.58) \quad \|k^S\|_{L_x^\infty L_t^2[0,T]} \lesssim \|F\|_{L_{x_1}^1 L_{\bar{x}_1}^{2,1} L_t^2[0,T]} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}.$$

The lemma is proved.  $\square$

To prepare our bootstrap arguments in the later section, similarly to the free case, we also consider the truncated versions of the above estimates.

By the same method we used to estimate

$$(3.59) \quad \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_c F(s) ds$$

along slanted lines, we obtain the following:

**Corollary 3.6.** *For  $|v| < 1$  and  $A > 0$ , suppose  $H = -\Delta + V$  has neither resonances nor eigenfunctions at 0. Let*

$$(3.60) \quad k_A(\cdot, t) := \int_0^{t-A} \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_c F(s) ds.$$

Then

$$(3.61) \quad \|k_A\|_{L_x^\infty L_t^2[A, \infty)} \lesssim \frac{1}{A} \left( \|F\|_{L_x^1 L_t^2} + \|F\|_{L_x^{\frac{3}{2}, 1} L_t^2} \right),$$

and for  $T > 0$ ,

$$(3.62) \quad \|k_A\|_{L_x^\infty L_t^2[A, T]} \lesssim \frac{1}{A} \left( \|F\|_{L_x^1 L_t^2[0, T]} + \|F\|_{L_x^{\frac{3}{2}, 1} L_t^2[0, T]} \right).$$

Define

$$(3.63) \quad k_A^S(x, t) := k_A(x + vt, t).$$

then

$$(3.64) \quad \|k_A^S(x, t)\|_{L_x^\infty L_t^2[A, \infty)} \lesssim \frac{1}{A} \left( \|F\|_{L_x^{\frac{3}{2}, 1} L_t^2} + \|F\|_{L_x^1 L_t^2} \right).$$

and for  $T > 0$ ,

$$(3.65) \quad \|k_A^S(x, t)\|_{L_x^\infty L_t^2[A, T]} \lesssim \frac{1}{A} \left( \|F\|_{L_x^{\frac{3}{2}, 1} L_t^2[0, T]} + \|F\|_{L_x^1 L_t^2[0, T]} \right).$$

Finally, in order to handle moving potentials, we consider some estimates with inhomogeneous terms along slanted lines:

Setting

$$(3.66) \quad F^S(x, t) = F(x + vt, t)$$

we have the following results.

**Lemma 3.7.** *Let  $A > 0$  and  $|\vec{\mu}| < 1$ ,  $|v| < 1$ . Suppose*

$$(3.67) \quad g_A(x, t) := \int_0^{t-A} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds,$$

$$(3.68) \quad g_A^S(x, t) = g_A(x + \vec{\mu}t, t).$$

We have

$$(3.69) \quad \|g_A(x, t)\|_{L_x^\infty L_t^2[A, \infty)} \lesssim \frac{1}{A} \|F^S\|_{L_x^1 L_t^2},$$

$$(3.70) \quad \|g_A^S(x, t)\|_{L_x^\infty L_t^2[A, \infty)} \lesssim \frac{1}{A} \|F^S\|_{L_x^1 L_t^2},$$

and for  $T > 0$ ,

$$(3.71) \quad \|g_A(x, t)\|_{L_x^\infty L_t^2[A, T]} \lesssim \frac{1}{A} \|F^S\|_{L_x^1 L_t^2[0, T]},$$



$$(3.72) \quad \|g_A^S(x, t)\|_{L_x^\infty L_t^2[A, T]} \lesssim \frac{1}{A} \|F^S\|_{L_x^1 L_t^2[0, T]}.$$

*Proof.* We know explicitly,

$$(3.73) \quad g_A(x, t) = \int_0^{t-A} \int_{|x-y|=t-s} \frac{F(y, s)}{|x-y|} dy ds.$$

Taking  $z = y - sv$ , we have

$$(3.74) \quad \begin{aligned} |g_A(x, t)| &= \left| \int_0^{t-A} \int_{|x-y|=t-s} \frac{F(y, s)}{|x-y|} dy ds \right| \\ &= \left| \int_0^{t-A} \int_{|x-z-vs|=t-s} \frac{F^S(z, s)}{|x-z-vs|} dz ds \right| \\ &\lesssim \int_0^{t-A} \int_{|m|=t-s} \frac{|F^S(x-vs-m, s)|}{|m|} dm ds \\ &\lesssim \int_0^{t-A} \int_{|m|=t-s} \frac{|F^S(x-v(t-|m|)-m, t-|m|)|}{|m|} dm ds \\ &\lesssim \frac{1}{A} \int |F^S(x-v(t-|m|)-m, t-|m|)| dm \end{aligned}$$

In the third line above, we applied a change of variable  $m = x - z - vs$  and in the fifth line, we again applied a change of variable  $v|m| + m = h$  with bounded Jacobian.

Therefore, if we set  $q = v(t - |m|) + m$ , we have

$$(3.75) \quad \begin{aligned} \|g(x, \cdot)\|_{L_t^2[A, \infty)} &\lesssim \frac{1}{A} \int \|F^S(x - q, \cdot)\|_{L_t^2} dq \\ &\lesssim \frac{1}{A} \|F^S\|_{L_x^1 L_t^2} \end{aligned}$$

The estimate for  $\|g_A^S(x, t)\|_{L_x^\infty L_t^2}$  is the same as we did for Corollary 3.4. Hence we obtain

$$(3.76) \quad \|g_A^S(x, t)\|_{L_x^\infty L_t^2[A, \infty)} \lesssim \frac{1}{A} \|F^S\|_{L_x^1 L_t^2}.$$

The the same as above, when we restrict to  $[0, T]$ , one has

$$(3.77) \quad \|g_A(x, t)\|_{L_x^\infty L_t^2[A, T]} \lesssim \frac{1}{A} \|F^S\|_{L_x^1 L_t^2[0, T]},$$

$$(3.78) \quad \|g_A^S(x, t)\|_{L_x^\infty L_t^2[A, T]} \lesssim \frac{1}{A} \|F^S\|_{L_x^1 L_t^2[0, T]}.$$

The lemma is proved.  $\square$

The above lemma can also be established by a duality argument. For the sake of completeness, we sketch the argument here. We only focus on

$$\|g_A(x, t)\|_{L_x^\infty L_t^2[A, T]} \lesssim \frac{1}{A} \|F^S\|_{L_x^1 L_t^2}.$$

Testing a function  $H(x, t) \in L_x^1 L_t^2$ , one has

$$\begin{aligned} \int_{\mathbb{R}^3} \int_A^T H(x, t) g_A(x, t) dt dx &= \int_{\mathbb{R}^3} \int_A^T H(x, t) \int_0^{t-A} \int_{|x-y|=t-s} \frac{F(y, s)}{|x-y|} \sigma(dy) ds dt dx \\ &= \int_{\mathbb{R}^3} \int_0^{T-A} F(y, s) \int_{s+A}^T \int_{|x-y|=t-s} \frac{H(y, t)}{|x-z-vs|} \sigma(dx) dt ds dy \\ &= \int_{\mathbb{R}^3} \int_0^{T-A} F^S(z, s) \int_{s+A}^T \int_{|x-y|=t-s} \frac{H(y, t)}{|x-z-vs|} \sigma(dx) dt ds dy. \end{aligned}$$

Then it suffices to show

$$(3.79) \quad \left\| \int_{s+A}^T \int_{|x-y|=t-s} \frac{H(y, t)}{|x-z-vs|} \sigma(dx) dt \right\|_{L_z^\infty L_s^2[0, T-A]} \lesssim \frac{1}{A} \|H\|_{L_x^1 L_t^2[0, T]}.$$

But with an almost identical argument as Corollary 3.4, the estimate (3.79) indeed holds, and therefore, our desired estimate holds too.

By [BecGo] or applying the structure formula of wave operators, with the calculations in the proof of Lemma 3.7, Corollary 3.6 and Theorems 3.2, 3.5, we have the perturbed version of the estimates (3.69) and (3.70). We omit the details here since the calculations are more or less identical.

**Theorem 3.8.** *For  $|\vec{\mu}| < 1, |v| < 1$  and  $A > 0$ , suppose  $H = -\Delta + V$  has neither resonances nor eigenfunctions at 0. Define*

$$(3.80) \quad k_A(x, t) := \int_0^{t-A} \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_c F(s) ds$$

$$(3.81) \quad k_A^S(x, t) = k_A(x + \vec{\mu}t, t).$$

We have

$$(3.82) \quad \|k_A(x, t)\|_{L_x^\infty L_t^2[A, \infty)} \lesssim \frac{1}{A} \left( \|F^S\|_{L_x^1 L_t^2} + \|F^S\|_{L_x^{\frac{3}{2}, 1} L_t^2} \right),$$

$$(3.83) \quad \|k_A^S(x, t)\|_{L_x^\infty L_t^2[A, \infty)} \lesssim \frac{1}{A} \left( \|F^S\|_{L_x^1 L_t^2} + \|F^S\|_{L_x^{\frac{3}{2}, 1} L_t^2} \right),$$

and for  $T > 0$ ,

$$(3.84) \quad \|k_A(x, t)\|_{L_x^\infty L_t^2[A, T]} \lesssim \frac{1}{A} \left( \|F^S\|_{L_x^1 L_t^2[0, T]} + \|F^S\|_{L_x^{\frac{3}{2}, 1} L_t^2[0, T]} \right),$$

$$(3.85) \quad \|k_A^S(x, t)\|_{L_x^\infty L_t^2[A, T]} \lesssim \frac{1}{A} \left( \|F^S\|_{L_x^1 L_t^2[0, T]} + \|F^S\|_{L_x^{\frac{3}{2}, 1} L_t^2[0, T]} \right).$$

By careful analysis and more complicated computations, one can extend all the results above to the linear Klein-Gordon equation, cf. [GC3].

#### 4. ENDPOINT REVERSED STRICHARTZ ESTIMATES

In this section, we show the endpoint reversed Strichartz estimates for the wave equation with charge transfer Hamiltonian. More precisely, we consider

$$(4.1) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x-vt)u = 0$$

with initial data

$$u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

Throughout this section, for simplicity, we furthermore assume  $V_i$  is compactly supported. With a little bit more careful analysis, one can easily obtain the same results for general case, see Remark 4.6.

Recall that after we apply the associated Lorentz transformation  $L$ , under the new coordinate, we denote

$$(4.2) \quad u_L(x'_1, x'_2, x'_3, t') = u(\gamma(x'_1 + vt'), x'_2, x'_3, \gamma(t' + vx'_1)),$$

and with the inverse transformation  $L^{-1}$

$$(4.3) \quad u(x, t) = u_L(\gamma(x - vt), x_2, x_3, \gamma(t - vx_1)).$$

Under the above setting, we state the main result of this section.

**Theorem 4.1.** *Let  $|v| < 1$ . Suppose  $u$  is a scattering state in the sense of Definition 1.2 and solves*

$$(4.4) \quad \partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

*with initial data*

$$(4.5) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then*

$$(4.6) \quad \sup_{x \in \mathbb{R}^3} \int_0^\infty |u(x, t)|^2 dt \lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

*Furthermore, if we denote*

$$(4.7) \quad u^S(x, t) := u(x + vt, t),$$

*then*

$$(4.8) \quad \sup_{x \in \mathbb{R}^3} \int_0^\infty |u^S(x, t)|^2 dt \lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

To show Theorem 4.1, we will apply a bootstrap process and decomposition into channels in the spirit of [RSS]. If there are no bound states, the bootstrap arguments simply work for the entire evolution. But in the presence of bound states, a more careful analysis is necessary. We will construct a truncated evolution and show that the estimates we obtain are independent of the truncation. Finally, we pass our estimates to the entire evolution.

**4.1. Bootstrap argument.** We set up the bootstrap argument and prove the initial assumptions for the bootstrap argument hold for big  $T$  with some positive constants.

By Duhamel's formula,

$$(4.9) \quad u(x, t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f + \cos(t\sqrt{-\Delta}) g - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} (V_1 + V_2(\cdot - vs)) u(s) ds.$$

By Grönwall's inequality, the endpoint reversed Strichartz estimates and the estimate along slanted lines for the free evolution, we have the following estimates as bootstrap assumptions.

**Lemma 4.2.** *For  $T > 0$  large, there exist constants  $C_1(T)$  and  $C_2(T)$  such that*

$$(4.10) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u(x, t)|^2 dt \leq C_1(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

*and if we denote*

$$u^S(x, t) = u(x + vt, t),$$

*then*

$$(4.11) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u^S(x, t)|^2 dt \leq C_2(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

*Proof.* To establish the bootstrap assumptions, we first notice that by the expression (4.9) and Grönwall inequality, we have

$$(4.12) \quad \int_{\mathbb{R}^3} |\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2 dx \lesssim e^{C|t|} (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

Clearly, estimates (4.10) and (4.11) hold for  $T = 0$ . Next, we note that for arbitrary  $T_0 > 0$ , from Theorem 2.2,

$$(4.13) \quad \begin{aligned} \sup_{x \in \mathbb{R}^3} \int_0^{T_0} |u(x, t)|^2 dt &\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \\ &+ C(T_0) \left( \sup_{x \in \mathbb{R}^3} \int_0^{T_0} |u(x, t)|^2 dt + \sup_{x \in \mathbb{R}^3} \int_0^{T_0} |u^S(x, t)|^2 dt \right) \end{aligned}$$

where  $C(T_0)$  can be computed explicitly, see Theorem 2.2 and duality argument as Lemma 3.7:

$$C(T_0) = \sup_{x \in \mathbb{R}^3} \int_{|x-y| \leq T_0} \frac{1}{|x-y|} |V_1| dy + \int_{|\hat{y}_1 - \hat{x}_1| \leq T_0} \int |V_2| dy_1 d\hat{y}_1.$$

We can perform a similar estimate for  $\sup_{x \in \mathbb{R}^3} \int_0^{T_0} |u^S(x, t)|^2 dt$ .

Therefore, for  $T_0$  small enough,

$$(4.14) \quad \sup_{x \in \mathbb{R}^3} \int_0^{T_0} |u(x, t)|^2 dt \leq C(T_0) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

$$(4.15) \quad \sup_{x \in \mathbb{R}^3} \int_0^{T_0} |u^S(x, t)|^2 dt \leq C(T_0) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

Iterating the above construction with the energy growth estimate (4.12), we can obtain that for  $T > 0$  large, there exists constant  $C_1(T)$ ,  $C_2(T)$  such that

$$(4.16) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u(x, t)|^2 dt \leq C_1(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2,$$

$$(4.17) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u^S(x, t)|^2 dt \leq C_2(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2,$$

as claimed □

Based on estimates (4.10), (4.11), we will run a bootstrap argument to improve these two estimate and reduce to estimates with constants independent of  $T$ .

We also have a perturbed version of Lemma 4.2 with the same constants in estimates (4.10) and (4.11) up to multiplication of a constant only depending on the potentials. Let

$$H_i = -\Delta + V_i, \quad i = 1, 2$$

and  $P_c(H_i)$  to be the projection onto the continuous spectrum of  $H_i$ .

**Lemma 4.3.** *For  $T > 0$  large, there exist constants  $C_1(T)$  and  $C_2(T)$  such that*

$$(4.18) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |P_c(H_1)u(x, t)|^2 dt \leq_{V_1} C_1(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2,$$

$$(4.19) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |P_c(H_2)u_L(x, t)|^2 dt \leq_{V_2} C_1(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

*Remark 4.4.* By symmetry, with  $C_1(T)$  and  $C_2(T)$ , we also have with  $T > 0$ ,

$$(4.20) \quad \sup_{x \in \mathbb{R}^3} \int_{-T}^0 |u(x, t)|^2 dt \leq C_1(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

and

$$(4.21) \quad \sup_{x \in \mathbb{R}^3} \int_{-T}^0 |u^S(x, t)|^2 dt \leq C_2(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

**4.2. Bound states.** Before we start the bootstrap analysis, it is necessary to understand the evolution of bound states.

In the following, for simplicity, we assume  $H_i - \Delta + V_i$ ,  $i = 1, 2$  has only one negative eigenvalue. With  $\lambda > 0$ ,  $\mu > 0$ ,

$$(4.22) \quad H_1 w = -\lambda^2 w, \quad H_2 m = -\mu^2 m.$$

$w$  and  $m$  decay exponentially by Agmon's estimate. The analysis can be easily adapted to the most general situation.

Set  $U(t, s)$  as evolution from  $s$  to  $t$  associated to the initial velocity and formally, we use  $\dot{U}(t, s)$  to denote the evolution associated the other initial data.

Suppose  $u(x, t)$  is a scattering state. We decompose the evolution as following,

$$(4.23) \quad u(x, t) = U(t, 0)f + \dot{U}(t, 0)g = a(t)w(x) + b(\gamma(t - vx_1))m_v(x, t) + r(x, t)$$

where

$$m_v(x, t) = m(\gamma(x_1 + vt), x_2, x_3).$$

With our decomposition, we know

$$(4.24) \quad P_c(H_1)r = r$$

and

$$(4.25) \quad P_c(H_2)r_L = r_L$$

where the Lorentz transformation  $L$  makes  $V_2$  stationary.

Surely, since  $u(x, t)$  is asymptotically orthogonal to the bound states of  $H_1$  and  $H_2$ , it forces  $a(t)$  to go to 0 and  $b(t)$  go to 0. Following the above construction, we do some preliminary calculations.

Plugging the evolution (4.23) into the equation (4.1) and taking inner product with  $w$ , we get

$$(4.26) \quad \begin{aligned} & \ddot{a}(t) - \lambda^2 a(t) + a(t) \langle V_2(x - vt)w, w \rangle \\ & + \langle V_2(x - vt)(b(\gamma(t - vx_1))m_v(x, t) + r(x, t)), w \rangle = 0. \end{aligned}$$

One can write

$$(4.27) \quad \ddot{a}(t) - \lambda^2 a(t) + a(t)c(t) + h(t) = 0,$$

where

$$(4.28) \quad c(t) := \langle V_2(x - vt)w, w \rangle$$

and

$$(4.29) \quad h(t) := \langle V_2(x - vt)(b(\gamma(t - vx_1))m_v(x, t) + r(x, t)), w \rangle.$$

Since  $w$  is exponentially localized by Agmon's estimate, we know

$$(4.30) \quad |c(t)| \lesssim e^{-\alpha|t|}.$$

The existence of the solution to the ODE (4.27) is clear. We study the long-time behavior of the solution. Write the equation as

$$(4.31) \quad \ddot{a}(t) - \lambda^2 a(t) = -[a(t)c(t) + h(t)],$$

and denote

$$(4.32) \quad N(t) := -[a(t)c(t) + h(t)].$$

Then

$$(4.33) \quad a(t) = \frac{e^{\lambda t}}{2} \left[ a(0) + \frac{1}{\lambda} \dot{a}(0) + \frac{1}{\lambda} \int_0^t e^{-\lambda s} N(s) ds \right] + R(t)$$

where

$$(4.34) \quad |R(t)| \lesssim e^{-\beta t},$$

for some positive constant  $\beta > 0$ . Therefore, the stability condition forces

$$(4.35) \quad a(0) + \frac{1}{\lambda} \dot{a}(0) + \frac{1}{\lambda} \int_0^\infty e^{-\lambda s} N(s) ds = 0.$$

Then under the stability condition (4.35),

$$(4.36) \quad a(t) = e^{-\lambda t} \left[ a(0) + \frac{1}{2\lambda} \int_0^\infty e^{-\lambda s} N(s) ds \right] + \frac{1}{2\lambda} \int_0^\infty e^{-\lambda|t-s|} N(s) ds.$$

We notice that in order to estimate  $a(t)$  and  $b(t)$ , we need a non-local term

$$(4.37) \quad \int_0^\infty e^{-\lambda s} N(s) ds,$$

and in all estimates, a global estimate for

$$(4.38) \quad \|b(\gamma(t - vx_1)) m_v(x, t) + r(x, t)\|_{L_x^\infty L_t^2[0, \infty)}$$

is involved. But for the general charge transfer model, a-priori, we do not have any global estimates. Therefore, we will consider a truncated version of the above construction restricted to interval  $t \in [0, T]$  for large positive  $T$ . Then one can run the bootstrap procedure for our truncated evolution.

For  $t \in [0, T]$ , we construct the following truncated version of the evolution:

$$u_T(x, t) = U(t, 0)f + \dot{U}(t, 0)g = a_T(t)w(x) + b_T(\gamma(t - vx_1)) m_v(x, t) + r_T(x, t).$$

For  $a_T(t)$ , we analyze the same ODE for  $a(t)$  again but restricted to  $[0, T]$  and instead of the stability condition

$$a(0) + \frac{1}{\lambda} \dot{a}(0) + \frac{1}{\lambda} \int_0^\infty e^{-\lambda s} N(s) ds = 0$$

we impose the condition that

$$a_T(0) + \frac{1}{\lambda} \dot{a}_T(0) + \frac{1}{\lambda} \int_0^T e^{-\lambda s} N(s) ds = 0.$$

The same construction can be applied to  $b_T$ .

**Lemma 4.5.** *From the construction above, we have the following estimates: for  $0 \ll A \ll T$ ,*

$$(4.39) \quad \|a_T\|_{L^\infty[0, T]} \lesssim \left( C(A, \lambda) + \frac{1}{\lambda A} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1}),$$

$$(4.40) \quad \|a_T\|_{L^1[0, T]} \lesssim \left( C(A, \lambda) + \frac{1}{\lambda A} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1}),$$

$$(4.41) \quad \|b_T\|_{L^\infty[0, T]} \lesssim \left( C(A, \mu) + \frac{1}{\mu A} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1}),$$

and

$$(4.42) \quad \|b_T\|_{L^1[0, T]} \lesssim \left( C(A, \mu) + \frac{1}{\mu A} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1}).$$

*Proof.* First of all, by the bootstrap assumption (4.18),

$$\|b_T(\gamma(t - vx_1))m_v(x, t) + r_T(x, t)\|_{L_x^\infty L_t^2[0, T]} \leq C_1(T)(\|f\|_{L^2} + \|g\|_{\dot{H}^1}).$$

For  $a_T(t)$ , we know that

$$(4.43) \quad \ddot{a}_T(t) - \lambda^2 a_T(t) + a_T(t) \langle V_2(x - vt)w, w \rangle + \langle V_2(x - vt)(b_T(\gamma(t - vx_1))m_v(x, t) + r_T(x, t)), w \rangle = 0.$$

We obtain

$$(4.44) \quad a_T(t) = \frac{e^{\lambda t}}{2} \left[ a_T(0) + \frac{1}{\lambda} \dot{a}_T(0) + \frac{1}{\lambda} \int_0^t e^{-\lambda s} N(s) ds \right] + R(t)$$

where

$$(4.45) \quad |R(t)| \lesssim e^{-\beta t},$$

With notations introduced above, we consider the truncated version of the stability condition,

$$(4.46) \quad a_T(0) + \frac{1}{\lambda} \dot{a}_T(0) + \frac{1}{\lambda} \int_0^T e^{-\lambda s} N(s) ds = 0.$$

So

$$(4.47) \quad a_T(t) = e^{-\lambda t} \left[ a_T(0) + \frac{1}{2\lambda} \int_0^T e^{-\lambda s} N(s) ds \right] + \frac{1}{2\lambda} \int_0^T e^{-\lambda|t-s|} N(s) ds.$$

where

$$(4.48) \quad N(t) = -[a_T(t)c(t) + h(t)]$$

with

$$(4.49) \quad |c(t)| \lesssim e^{-\alpha|t|}$$

$$(4.50) \quad h(t) := \langle V_2(x - vt)[b_T(t - vx_1)m_v(x, t) + r_T(x, t)], w \rangle.$$

For  $0 \ll A \ll T$  fixed, we can always bound the  $L^\infty$  norm of  $a_T$  on the interval  $[0, A]$  by Grönwall's inequality. Therefore, it suffices to estimate the  $L^\infty$  norm of  $a_T$  from  $A$  to  $T$ . Note that  $|c(t)| \lesssim e^{-\alpha|t|}$ , for  $A$  large, one can always absorb the effects from  $\int_A^T a_T(t)c(t) dt$  into the left-hand side. Hence it reduces to estimate the  $L_t^1$  norm of  $h(t)$  restricted to  $[A, T]$ .

Consider the integral

$$\int_A^T |h(t)| dt = \int_A^T |\langle V_2(x - vt)[b_T(\gamma(t - vx_1))m_v(x, t) + r_T(x, t)], w \rangle| dt.$$

Clearly,

$$\begin{aligned} & \int_A^T |V_2(x - vt)[b_T(\gamma(t - vx_1))m_v(x, t) + r_T(x, t)]| dt \lesssim \\ & \left( \int_A^T |(b_T(\gamma(t - vx_1))m_v(x, t) + r_T(x, t))|^2 dt \right)^{\frac{1}{2}} \left( \int_A^T |V_2(x - vt)|^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Note that

$$(4.51) \quad \left| \left\langle \left( \int_A^T |V_2(\cdot - vt)|^2 dt \right)^{\frac{1}{2}}, w \right\rangle \right| \lesssim \frac{1}{A}.$$

By the preliminary calculations above, we can estimate the  $L^\infty$  norm of  $a_T(t)$ ,

$$\begin{aligned}
\|a_T\|_{L^\infty[0,T]} &\lesssim C(A, \lambda) (\|f\|_{L^2} + \|g\|_{\dot{H}^1}) + \frac{1}{\lambda} \int_A^T |h(t)| dt \\
&\lesssim C(A, \lambda) (\|f\|_{L^2} + \|g\|_{\dot{H}^1}) \\
&\quad + \frac{1}{\lambda A} \left( \int_A^T |(b_T(\gamma(t - vx_1)) m_v(x, t) + r_T(x, t))|^2 dt \right)^{\frac{1}{2}} \\
&\lesssim \left( C(A, \lambda) + \frac{1}{\lambda A} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1}).
\end{aligned}
\tag{4.52}$$

Similarly, for the  $L^1$  norm of  $a_T(t)$ ,

$$\|a_T\|_{L^1[0,T]} \lesssim \left( C(A, \lambda) + \frac{1}{\lambda A} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1}).
\tag{4.53}$$

After applying a Lorentz transformation, we have analogous estimates for  $b_T(t)$ :

$$\|b_T\|_{L^\infty[0,T]} \lesssim \left( C(A, \mu) + \frac{1}{\mu A} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1}),
\tag{4.54}$$

$$\|b_T\|_{L^1[0,T]} \lesssim \left( C(A, \mu) + \frac{1}{\mu A} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1}).
\tag{4.55}$$

The lemma is proved. □

In the following subsections, we will show estimates with constants independent of  $T$ ,

$$\sup_{x \in \mathbb{R}^3} \int_0^T |u_T(x, t)|^2 dt \leq C_1 (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2
\tag{4.56}$$

$$\sup_{x \in \mathbb{R}^3} \int_0^T |u_T^S(x, t)|^2 dt \leq C_2 (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.
\tag{4.57}$$

Then we know our construction of  $u_T$  has estimates independent of  $T$ . As  $T \rightarrow \infty$ , the stability condition (4.35) will be recovered from

$$a_T(0) + \frac{1}{\lambda} \dot{a}_T(0) + \frac{1}{\lambda} \int_0^T e^{-\lambda s} N(s) ds = 0.
\tag{4.58}$$

and

$$a_T(t) = e^{-\lambda t} \left[ a_T(0) + \frac{1}{2\lambda} \int_0^T e^{-\lambda s} N(s) ds \right] + \frac{1}{2\lambda} \int_0^T e^{-\lambda|t-s|} N(s) ds.
\tag{4.59}$$

Therefore, from the estimates for  $u_T$ , we can obtain the desired estimates for a scattering state  $u(x, t)$ ,

$$\sup_{x \in \mathbb{R}^3} \int_0^\infty |u(x, t)|^2 dt \leq C_1 (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2
\tag{4.60}$$

$$\sup_{x \in \mathbb{R}^3} \int_0^\infty |u^S(x, t)|^2 dt \leq C_2 (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.
\tag{4.61}$$

Therefore in the remaining part of this section, we will analyze the bootstrap process for  $u_T(x, t)$  carefully.



**4.3. Decomposition into channels.** Following the notations above, for  $t \in [0, T]$ , consider

$$u_T(x, t) = U(t, 0)f + \dot{U}(t, 0)g = a_T(t)w(x) + b_T(\gamma(t - vx_1))m_v(x, t) + r_T(x, t).$$

There exist constants  $C_1(T)$  and  $C_2(T)$  such that

$$(4.62) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u_T(x, t)|^2 dt \leq C_1(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

and

$$(4.63) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u_T^S(x, t)|^2 dt \leq C_2(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

We run our bootstrap argument for  $u_T(x, t)$ . Notice that since  $V_i$ ,  $i = 1, 2$  is a short-range potential and  $V_2(x - vt)$  moves away from  $V_1$ , intuitively,  $u(x, t)$  will have different dominant behaviors in different regions in  $\mathbb{R}^3$ . To make this heuristic rigorous, we perform a decomposition of channels based on it. For some fixed small  $\delta > 0$ , we introduce a partition of unity associated with the sets

$$(4.64) \quad B_{\delta t}(0) = \{x : |x| \leq \delta t\}, \quad B_{\delta t}(tv) = \{x : |x - (tv, 0, 0)| \leq \delta t\}$$

and

$$(4.65) \quad \mathbb{R}^3 \setminus (B_{\delta t}(0) \cup B_{\delta t}(tv)).$$

To be more precisely, let  $\chi_1(x, t)$  be a smooth cutoff function such that

$$(4.66) \quad \chi_1(x, t) = 1, \forall x \in B_{\delta t}(0), \quad \chi_1(x, t) = 0, \forall x \in \mathbb{R}^3 \setminus B_{2\delta t}(0).$$

One might assume  $t \geq t_0$  for some large  $t_0$ . We also define

$$(4.67) \quad \chi_2(x, t) = \chi_1(x - vt, t), \quad \chi_3 = 1 - \chi_1 - \chi_2.$$

Note that we only consider the estimates for large  $t$ , so one might also assume the support of  $\chi_1(x, t)$  contains the support of  $V_1(x)$  and support of  $\chi_2(x, t)$  contains the support of  $V_2(\cdot - vt)$ .

With the partition above, we rewrite the evolution as

$$(4.68) \quad u_T(x, t) = \chi_1(x, t)u_T(x, t) + \chi_2(x, t)u_T(x, t) + \chi_3(x, t)u_T(x, t).$$

We will discuss  $\chi_i(x, t)u_T(x, t)$ ,  $i = 1, 2, 3$ , separately.

Based on Duhamel's formula, we will compare  $u$  to different evolution groups on different "channels".

For

$$(4.69) \quad \chi_1(x, t)u_T(x, t),$$

we will compare it to

$$(4.70) \quad W_1(t)f + \dot{W}_1(t)g$$

where

$$(4.71) \quad W_1(t) := \frac{\sin(t\sqrt{H_1})}{\sqrt{H_1}}.$$

As to

$$(4.72) \quad \chi_2(x, t)u_T(x, t),$$

it will be compared to

$$(4.73) \quad W_2(t)f + \dot{W}_2(t)g$$

where  $W_2(t, s)$  denotes the evolution associated with the Hamiltonian  $-\Delta + V_2(x - vt)$  and initial velocity  $f$ , starting from  $s$  to  $t$ . And formally,  $\dot{W}_2(t, s)$  is used to denote the evolution associated with  $g$  from  $s$  to  $t$ . Here the dot in  $\dot{W}_2$  is not the time derivative but simply a notation. These evolutions can be obtained from the entries of the solution map if we write the wave equation  $\partial_{tt}u - \Delta u + V_2(x - vt)u = 0$  using

the Hamiltonian structure. We also use the short-hand notation  $W_2(t)$  and  $\dot{W}_2(t)$  to denote the evolutions starting at  $s = 0$ .

Finally

$$(4.74) \quad \chi_3(x, t)u_T(x, t)$$

is compared with

$$(4.75) \quad W_0(t)f + \dot{W}_0(t)g$$

where

$$(4.76) \quad W_0(t) := \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}.$$

To be more explicit, we write

$$(4.77) \quad \begin{aligned} \chi_1(x, t)u_T(x, t) &= \chi_1(x, t)W_1(t)f + \chi_1(x, t)\dot{W}_1(t)g \\ &\quad - \chi_1(x, t) \int_0^t W_1(t-s)V_2(\cdot - sv)u_T(s) ds, \end{aligned}$$

$$(4.78) \quad \begin{aligned} \chi_2(x, t)u_T(x, t) &= \chi_2(x, t)W_2(t)f + \chi_2(x, t)\dot{W}_2(t)g \\ &\quad - \chi_2(x, t) \int_0^t W_2(t,s)V_1u_T(s) ds \end{aligned}$$

and

$$(4.79) \quad \begin{aligned} \chi_3(x, t)u_T(x, t) &= \chi_3(x, t)W_0(t)f + \chi_3(x, t)\dot{W}_0(t)g \\ &\quad - \chi_3(x, t) \int_0^t W_0(t-s)(V_1 + V_2(\cdot - vs))u_T(s) ds. \end{aligned}$$

**4.4. Analysis of the three channels.** We will use the notations

$$(4.80) \quad \begin{aligned} u_T(x, t) &= a_T(t)w(x) + b_T(\gamma(t - vx_1))m_v(x, t) + r_T(x, t) \\ &=: a_T(t)w(x) + u_{T,1}(x, t) \\ &=: b_T(\gamma(t - vx_1))m_v(x, t) + u_{T,2}(x, t). \end{aligned}$$

Note that

$$(4.81) \quad P_c(H_1)(u_{T,1}) = u_{T,1}$$

and

$$(4.82) \quad P_c(H_2)(u_{T,2})_L = (u_{T,2})_L.$$

The free channel and the channel associated with  $H_1$  are easy to analyze with the endpoint reversed Strichartz estimate and results for estimates along slanted lines, Theorems 2.2, 2.3, 3.2 and Lemma 3.1.

**4.4.1. Analysis of  $\chi_1(x, t)u_T(x, t)$ :** We consider

$$(4.83) \quad \begin{aligned} \chi_1(x, t)u_T(x, t) &= \chi_1(x, t)W_1(t)f + \chi_1(x, t)\dot{W}_1(t)g \\ &\quad - \chi_1(x, t) \int_0^t W_1(t-s)V_2(\cdot - sv)u_T(s) ds. \end{aligned}$$

Given  $B$  fixed and  $0 \ll B \ll T$ , one can always bound the integrals restricted to  $[0, B]$ ,

$$\int_0^B |\chi_1(x, t)u_T(x, t)|^2 dt, \quad \int_0^B |\chi_1(x, t)u_T^S(x, t)|^2 dt$$

by a prescribed constant by Grönwall's inequality as Lemma 4.2. Therefore, it suffices to consider the integrals over  $[B, T]$ . If we fixed  $0 \ll A \ll T$  large, one can always find a big constant  $B$  such that

$A \ll \frac{(v-2\delta)}{1+v}B$ . Then when we consider the integrals from  $B$  to  $T$ , by the finite speed of propagation and the fact that  $V_2$  is compactly supported, we can further reduce

$$(4.84) \quad \begin{aligned} \chi_1(x, t)u_T(x, t) &= \chi_1(x, t)W_1(t)f + \chi_1(x, t)\dot{W}_1(t)g \\ &\quad - \chi_1(x, t) \int_0^{t-A} W_1(t-s)V_2(\cdot-sv)u_T(s) ds. \end{aligned}$$

For  $s > t - A$ , the center of  $V_2$  is of distance at least  $|(t - A)v|$  away from the center of the support of  $\chi_1$ . Meanwhile,  $t - s$  is at most  $A$ . So the effects caused by  $W_1(t-s)V_2(\cdot-sv)u_T(s)$  will not influence the points in the support of  $\chi_1$ .

First, we consider the endpoint reversed Strichartz estimate (4.10),

$$\begin{aligned} \int_0^T |\chi_1(x, t)u_{T,1}(x, t)|^2 dt &\lesssim \int_0^T \left| \chi_1(x, t)W_1(t)P_c(H_1)f + \chi_1(x, t)\dot{W}_1(t)P_c(H_1)g \right|^2 dt \\ &\quad + \int_0^T \left| \chi_1(x, t) \int_0^{t-A} W_1(t-s)P_c(H_1)V_2(\cdot-sv)u_T(s) ds \right|^2 dt \\ &\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \\ &\quad + \int_0^B \left| \chi_1(x, t) \int_0^t W_1(t-s)P_c(H_1)V_2(\cdot-sv)u_T(s) ds \right|^2 dt \\ &\quad + \int_B^T \left| \chi_1(x, t) \int_0^{t-A} W_1(t-s)P_c(H_1)V_2(\cdot-sv)u_T(s) ds \right|^2 dt \\ &\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 + C(B)(\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \\ &\quad + \int_B^T \left| \chi_1(x, t) \int_0^{t-A} W_1(t-s)P_c(H_1)V_2(\cdot-sv)u_T(s) ds \right|^2 dt \\ &\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 + C(B)(\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \\ &\quad + \frac{1}{A}C_2(T)(\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2. \end{aligned}$$

In the above calculations, for the second inequality, we applied the endpoint Strichartz estimate for perturbed wave equations, cf. Theorem 2.3:

$$\int_0^T \left| \chi_1(x, t)W_1(t)P_c(H_1)f + \chi_1(x, t)\dot{W}_1(t)P_c(H_1)g \right|^2 dt \lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

For the third inequality, we used the fact that  $B$  is a fixed big constant, one can always find  $C(B)$  independent of  $T$  to ensure the inequality holds as we did in Lemma 4.3:

$$\int_0^B \left| \chi_1(x, t) \int_0^t W_1(t-s)P_c(H_1)V_2(\cdot-sv)u_T(s) ds \right|^2 dt \lesssim C(B)(\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

For the last inequality, we used the bootstrap assumption (4.11) and the results from the section on estimates along slanted lines, Theorem 3.8 and Corollary 3.6. By Theorem 3.8,

$$\begin{aligned} \int_B^T \left| \chi_1(x, t) \int_0^{t-A} W_1(t-s)P_c(H_1)V_2(\cdot-sv)u_T(s) ds \right|^2 dt &\lesssim \frac{1}{A^2} \|V_2\|_{L_x^2}^2 \sup_x \int_0^T |u_T(t)|^2 dt \\ &\lesssim \frac{1}{A} C_2(T)(\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2. \end{aligned}$$

Therefore,

$$(4.85) \quad \int_0^T |\chi_1(x, t)u_{T,1}(x, t)|^2 dt \lesssim \left( C_0 + C(A) + \frac{1}{A}C_2(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

For the remaining piece, by estimates (4.39), (4.40) and Agmon's estimate,

$$(4.86) \quad \int_0^T |\chi_1(x, t) a_T(t) w(x)|^2 dt \lesssim \left( C(A, \lambda) + \frac{1}{\lambda A} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

Therefore, with estimates (4.85) and (4.86), for the endpoint reversed estimate, we obtain

$$(4.87) \quad C_1(T) \lesssim C_0 + C(A, B) + \frac{1}{A} C_2(T)$$

in the first channel. So for  $A$  large, in this channel, we have the condition for the bootstrap argument.

Next we consider the estimate along the slanted line  $(x + vt, t)$ .

Denoting

$$(4.88) \quad u_{T,1}^S(x, t) = \chi_1(x + vt, t) u_{T,1}(x + vt, t),$$

we want to estimate

$$(4.89) \quad \int_0^T |\chi_1(x + vt, t) u_{T,1}(x + vt, t)|^2 dt = \int_0^T |u_{T,1}^S(x, t)|^2 dt.$$

Furthermore, we introduce

$$(4.90) \quad D_1^S(x, t) := D_1(x + vt, t)$$

where

$$(4.91) \quad D_1(x, t) := \chi_1(x, t) W_1(t) P_c(H_1) f + \chi_1(x, t) \dot{W}_1(t) P_c(H_1) g;$$

$$(4.92) \quad k_1^S(x, t) := k_1(x + vt, t)$$

where

$$(4.93) \quad k_1(x, t) := \chi_1(x, t) \int_0^t W_1(t-s) P_c(H_1) V_2(\cdot - sv) u_T(s) ds;$$

$$(4.94) \quad E_1^S(x, t) := E_1(x + vt, t)$$

where

$$(4.95) \quad E_1(x, t) := \chi_1(x, t) \int_0^{t-A} W_1(t-s) P_c(H_1) V_2(\cdot - sv) u_T(s) ds.$$

Then we can conclude

$$(4.96) \quad \begin{aligned} \int_0^T |u_{T,1}^S|^2 dt &\lesssim \int_0^T |D_1^S|^2 dt + \int_0^B |k_1^S|^2 dt + \int_B^T |E_1^S|^2 dt \\ &\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 + C(B) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \\ &\quad + \frac{1}{A} C_2(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \end{aligned}$$

similar to the analysis of estimate (4.85) via Theorems 3.2, 3.8 and Corollary 3.6.

For the piece with bound states, by estimate (4.40) and Agmon's estimate,

$$(4.97) \quad \begin{aligned} &\int_0^T |\chi_1(x + vt, t) a_T(t) w(x + vt)|^2 dt \\ &\lesssim \left( C(A, \lambda) + \frac{1}{\lambda A} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2. \end{aligned}$$

Therefore, with estimates (4.96) and (4.97), we obtain

$$(4.98) \quad C_2(T) \lesssim C_0 + C(A, B) + \frac{1}{A} C_2(T)$$

in the first channel. So for  $A$  large, in this channel, we obtain the desired reduction for the bootstrap argument.

4.4.2. *Analysis of  $\chi_2(x, t)u_T(x, t)$ :* Now we consider the most delicate channel which is the channel associated to the moving potential.

$$(4.99) \quad \begin{aligned} \chi_2(x, t)u_T(x, t) &= \chi_2(x, t)W_2(t)f + \chi_2(x, t)\dot{W}_2(t)g \\ &\quad - \chi_2(x, t) \int_0^t W_2(t, s)V_1 u_T(s) ds. \end{aligned}$$

Again, by the finite speed of propagation, it suffices to consider

$$(4.100) \quad \begin{aligned} \chi_2(x, t)u_T(x, t) &= \chi_2(x, t)W_2(t)f + \chi_2(x, t)\dot{W}_2(t)g \\ &\quad - \chi_2(x, t) \int_0^{t-A} W_2(t, s)V_1 u_T(s) ds. \end{aligned}$$

Note that with the Lorentz transformation associated with  $V_2(x - vt)$ , we have

$$(4.101) \quad (u_T)_L(x'_1, x'_2, x'_3, t') = u_T(\gamma(x'_1 + vt'), x'_2, x'_3, \gamma(t' + vx'_1))$$

and

$$(4.102) \quad u_T(x, t) = (u_T)_L(\gamma(x_1 - vt), x_2, x_3, \gamma(t - vx_1)).$$

The endpoint reversed Strichartz estimate (4.10) for this channel is equivalent to the estimate along the slanted line  $(x - vt, t)$  under the new frame. Meanwhile, the estimate along the slanted line  $(x + vt, t)$ , see (4.11), for this channel is equivalent to the endpoint reversed Strichartz estimate with respect to the new frame.

Denote  $\tilde{f}$  and  $\tilde{g}$  to denote the initial data with respect to this new frame under which  $V_2$  is stationary and  $V_1$  is moving. We use  $W_2^L(t)$  and  $\dot{W}_2^L(t)$  to denote the evolutions associated to  $\tilde{f}$  and  $\tilde{g}$  respectively in the new frame. By construction, in the new frame,  $W_2^L(t)$  is the sine evolution with respect to  $H_2$ . By Theorem 2.8, we know

$$(4.103) \quad (\|\tilde{f}\|_{L^2} + \|\tilde{g}\|_{\dot{H}^1}) \simeq (\|f\|_{L^2} + \|g\|_{\dot{H}^1}).$$

Denote

$$(4.104) \quad D_2^S(x, t) := D_2(x - vt, t)$$

where

$$(4.105) \quad D_2(x, t) := W_2^L(t)P_c(H_2)\tilde{f} + \dot{W}_2^L(t)P_c(H_2)\tilde{g};$$

$$(4.106) \quad k_2^S(x, t) := k_2(x - vt, t)$$

where

$$(4.107) \quad k_2(x, t) := \int_0^t W_2^L(t-s)P_c(H_2)V_1(s)u_T(s) ds;$$

$$(4.108) \quad E_2^S(x, t) := E_2(x - vt, t)$$

where

$$(4.109) \quad E_2(x, t) = \int_0^{t-A} W_2^L(t-s)P_c(H_2)V_1(s)u_T(s) ds.$$

With the estimates along the slanted line  $(x - vt, t)$  for  $W_2^L(t)$ , Theorem 3.2, we know

$$\begin{aligned}
\int_0^T |\chi_2(x, t) u_{T,2}(x, t)|^2 dt &= \int_0^T |(u_{T,2})_L(\gamma(x_1 - vt), x_2, x_3, \gamma(t - vx_1))|^2 dt \\
&\lesssim \int_0^T |D_2^S|^2 dt + \int_0^B |k_2^S|^2 dt + \int_B^T |E_2^S|^2 dt \\
&\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 + C(B) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \\
&\quad + \frac{1}{A} \left( \|V_1 u_T\|_{L_x^1 L_t^2[0, T]} \right)^2 \\
&\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 + C(B) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \\
&\quad + \frac{1}{A} \left( \|u_T\|_{L_x^\infty L_t^2[0, T]} \right)^2 \\
&\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 + C(B) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \\
&\quad + \frac{1}{A} C_1(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2
\end{aligned} \tag{4.110}$$

by the bootstrap assumption (4.10), Theorem 3.8 and Corollary 3.6. For the third inequality, we also use the fact  $A$  is a fixed big constant, we can always find  $C(A)$  independent of  $T$  to ensure the inequality holds.

For the piece with bound states, by estimate (4.41) and Agmon's estimate, one has

$$\begin{aligned}
\int_0^T |\chi_2(x, t) b_T(\gamma(t - vx_1)) m_v(x, t)|^2 dt \\
\lesssim \left( C(A, \mu) + \frac{1}{\mu A} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.
\end{aligned} \tag{4.111}$$

Hence in this channel, with estimates (4.110) and (4.111),

$$C_1(T) \lesssim C_0 + C(A, B) + \frac{1}{A} C_1(T). \tag{4.112}$$

So for  $A$  large, in this channel, we achieve the condition for the bootstrap argument.

Now we analyze the estimate along  $(x + vt, t)$ . The argument here is similar to the analysis for the first channel.

Denote

$$u_{T,2}^S(x, t) := \chi_2(x + vt, t) u_{T,2}(x + vt, t). \tag{4.113}$$

Then

$$\begin{aligned}
\int_0^T |u_{T,2}^S(x, t)|^2 dt &\lesssim \int_0^T |(u_{T,2})_L(x, t)|^2 dt \\
&\lesssim \int_0^T |D_2(x, t)|^2 dt + \int_0^T |k_2(x, t)|^2 dt + \int_0^T |E_2(x, t)|^2 dt \\
&\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 + C(B) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \\
&\quad + \frac{1}{A} \left( \|V_1 u_T\|_{L_x^1 L_t^2(0, T)} \right)^2 \\
&\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 + C(B) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \\
&\quad + \frac{1}{A} C_1(T) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2
\end{aligned} \tag{4.114}$$

with the bootstrap assumption (4.10) and Corollary 3.6.

For the remaining piece with bound states, by a similar argument to estimate (4.86), we have

$$(4.115) \quad \int_0^T |\chi_2(x+vt, t) (b_T(\gamma((1-v^2)t - vx_1)) m_v(x+vt, t))|^2 dt \\ \lesssim \left( C(A, \mu) + \frac{1}{\mu A} C_1(T) \right) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

Therefore, in this channel, we obtain

$$(4.116) \quad C_2(T) \lesssim C_0 + C(A, B) + \frac{1}{A} C_1(T).$$

For  $A$  large, in this channel, we recapture the condition for the bootstrap argument.

4.4.3. *Analysis of  $\chi_3(x, t)u_T(x, t)$ :* Finally, we consider the free channel  $\chi_3(x, t)u_T(x, t)$ . In this channel, we can estimate all pieces together since the dominant evolution is the free ones.

We know

$$(4.117) \quad \chi_3(x, t)u_T(x, t) = \chi_3(x, t)W_0(t)f + \chi_3(x, t)\dot{W}_0(t)g \\ - \chi_3(x, t) \int_0^t W_0(t-s)(V_1 + V_2(\cdot - vs))u_T(s) ds.$$

By the finite speed of propagation as above, it suffices to consider

$$(4.118) \quad \chi_3(x, t)u_T(x, t) = \chi_3(x, t)W_0(t)f + \chi_3(x, t)\dot{W}_0(t)g \\ - \chi_3(x, t) \int_0^{t-A} W_0(t-s)(V_1 + V_2(\cdot - vs))u_T(s) ds.$$

Consider the endpoint reversed Strichartz estimate,

$$(4.119) \quad \int_0^T |\chi_3(x, t)u_T(x, t)|^2 dt \lesssim \int_0^T \left| \chi_3(x, t)W_0(t)f + \chi_3(x, t)\dot{W}_0(t)g \right|^2 dt \\ + \int_0^T \left| \chi_3(x, t) \int_0^{t-A} W_0(t-s)V_1 u_T(s) ds \right|^2 dt \\ + \int_0^T \left| \chi_3(x, t) \int_0^{t-A} W_0(t-s)V_2(\cdot - sv)u_T(s) ds \right|^2 dt \\ \lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 + C(B) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \\ + \frac{1}{A} (C_1(T) + C_2(T)) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

For the last inequality, we apply the bootstrap assumptions (4.10) and (4.11).

$$(4.120) \quad C_1(T) \lesssim C_0 + C(A, B) + \frac{1}{A} (C_1(T) + C_2(T)).$$

So for  $A$  large, in this channel, we obtain the condition for bootstrap argument.

Next we consider the estimate along slanted line  $(x+vt, t)$ .

Denote

$$(4.121) \quad u_T^S(x, t) = \chi_3(x+vt, t)u_T(x+vt, t),$$

$$(4.122) \quad u_{T,3}^S(x, t) := u_{T,3}(x+vt, t)$$

where

$$(4.123) \quad u_{T,3}(x, t) := \chi_3(x, t)W_0(t)f + \chi_3(x, t)\dot{W}_0(t)g;$$

$$(4.124) \quad k_3^S(x, t) := k_3(x+vt, t)$$

where

$$(4.125) \quad k_3(x, t) := \chi_3(x, t) \int_0^t W_0(t-s) (V_1 + V_2(\cdot - sv)) u_T(s) ds;$$

$$(4.126) \quad E_3^S(x, t) := E_3(x + vt, t)$$

where

$$(4.127) \quad E_3(x, t) := \chi_3(x, t) \int_0^{t-A} W_0(t-s) (V_1 + V_2(\cdot - sv)) u_T(s) ds.$$

Then

$$(4.128) \quad \begin{aligned} \int_0^T |u_T^S(x, t)|^2 dt &\lesssim \int_0^T |u_{T,3}^S(x, t)|^2 dt + \int_0^B |k_3^S(x, t)|^2 dt + \int_B^T |E_3^S(x, t)|^2 dt \\ &\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 + C(B) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \\ &\quad + \frac{1}{A} (C_1(T) + C_2(T)) (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \end{aligned}$$

Therefore, we obtain

$$(4.129) \quad C_2(T) \lesssim C_0 + C(A, B) + \frac{1}{A} (C_1(T) + C_2(T))$$

along the free channel. So for  $A$  large, we recapture the condition for the bootstrap argument.

**4.5. Conclusion.** Finally, by the results from the analysis of three channels above, we conclude

$$(4.130) \quad C_1(T) \lesssim C_0 + \frac{1}{A} C_1(T) + \frac{1}{A} C_2(T) + C(A, B)$$

$$(4.131) \quad C_2(T) \lesssim C_0 + \frac{1}{A} C_1(T) + \frac{1}{A} C_2(T) + C(A, B)$$

where  $C_1(T)$  is the constant appearing for the bootstrap assumption (4.10) for the endpoint reversed Strichartz estimate and  $C_2(T)$  is the constant for the bootstrap assumption (4.11) for the estimate along  $(x + vt, t)$ .

We apply the bootstrap argument for these two estimates simultaneously. We conclude that  $C_1(T)$  and  $C_2(T)$  are independent of  $T$ . In other words, one has

$$(4.132) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u_T(x, t)|^2 dt \leq C_1 (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

$$(4.133) \quad \sup_{x \in \mathbb{R}^3} \int_0^T |u_T^S(x, t)|^2 dt \leq C_2 (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

Finally, as we discussed above, passing  $T$  to  $\infty$ , we will recover those two estimates for a scattering state  $u(x, t)$ :

$$(4.134) \quad \sup_{x \in \mathbb{R}^3} \int_0^\infty |u(x, t)|^2 dt \lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

and

$$(4.135) \quad \sup_{x \in \mathbb{R}^3} \int_0^\infty |u^S(x, t)|^2 dt \lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$



*Remark 4.6.* In the above analysis, we assumed  $V_1$  and  $V_2$  are compactly supported. With more careful calculations, it is easy to extend the above results to  $V_1$  and  $V_2$  decay as we assume in the Definition 1.1. In this case, instead of vanishing, the smallness conditions of our bootstrap procedure are from the smallness of tails of  $V_1$  and  $V_2$  in  $L_x^1$  and  $L_x^{\frac{3}{2},1}$  in the estimates for the following terms

$$(4.136) \quad \chi_1 \int_{t-A}^t W_1(t-s) V_2(\cdot - sv) u_T(s) ds,$$

$$(4.137) \quad \chi_2 \int_{t-A}^t W_2^L(t-s) V_1(s) u_T(s) ds$$

and

$$(4.138) \quad \chi_3 \int_{t-A}^t W_0(t-s) (V_1 + V_2(\cdot - vs)) u_T(s) ds.$$

To demonstrate, we compute a concrete example below.

$$(4.139) \quad \begin{aligned} & \left\| \int_{t-A}^t W_0(t-s) V_2(\cdot - sv) u_T(s) ds \right\|_{L_t^2[A,T]} \Big|_{x=0} \\ &= \left\| \int_{|y| \leq A} \frac{1}{|y|} V_2(y - v(t - |y|)) u_T((t - |y|)) dy \right\|_{L_t^2[B,T]} \\ &\lesssim \left( \frac{A^2}{\langle A \rangle^\alpha} \right) \sup_x \|u_T\|_{L_t^2[0,T]} \\ &\lesssim \frac{1}{A} \sup_x \|u_T\|_{L_t^2[0,T]}. \end{aligned}$$

All other terms can be estimated by a similar way.

## 5. STRICHARTZ ESTIMATES AND ENERGY BOUND

We know from the introduction that weighted estimates play important roles in building Strichartz estimates. In this section, we establish weighted estimates for a scattering state to the wave equation with charge transfer Hamiltonian. Just for the sake of convenience, we will restate our main theorems in this section.

Throughout this subsection, we will use the short-hand notation

$$(5.1) \quad L_t^p L_x^q := L_t^p([0, \infty), L_x^q).$$

**Corollary 5.1.** *Let  $|v| < 1$ . Suppose  $u$  is a scattering state in the sense of Definition 1.2 and that it solves*

$$(5.2) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

*with initial data*

$$(5.3) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x)$$

*Then for  $\alpha > 3$ ,*

$$(5.4) \quad \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^\alpha} u^2(x, t) dx dt \lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

*and*

$$(5.5) \quad \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x - vt \rangle^\alpha} u^2(x, t) dx dt \lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

*Proof.* The two weighted estimates above follow easily from Theorem 4.1.

For the first one,

$$(5.6) \quad \begin{aligned} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^\alpha} u^2(x, t) dx dt &\lesssim \left( \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^\alpha} dx \right) \sup_x \int_{\mathbb{R}^+} u^2(x, t) dt \\ &\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \end{aligned}$$

by the endpoint reversed Strichartz estimate (4.6) for  $u$ .

For the second one, one has

$$(5.7) \quad \begin{aligned} \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x - vt \rangle^\alpha} u^2(x, t) dx dt &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{1}{\langle y \rangle^\alpha} u^2(y + vt, t) dy dt \\ &\lesssim \left( \int_{\mathbb{R}^3} \frac{1}{\langle y \rangle^\alpha} dy \right) \sup_x \int_{\mathbb{R}^+} |u^S(x, t)|^2 dt \\ &\lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2 \end{aligned}$$

by our estimate (4.8) along the slanted line  $(x + vt, t)$ .

We are done. □

**Theorem 5.2.** *Let  $|v| < 1$ . Suppose  $u$  is a scattering state in the sense of Definition 1.2 which solves*

$$(5.8) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

*with initial data*

$$(5.9) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then for  $p > 2$ , and  $(p, q)$  satisfying*

$$(5.10) \quad \frac{1}{2} = \frac{1}{p} + \frac{3}{q}$$

*we have*

$$(5.11) \quad \|u\|_{L_t^p([0, \infty), L_x^q)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}$$

*Proof.* Following [LSch], we set  $A = \sqrt{-\Delta}$  and notice that

$$(5.12) \quad \|Af\|_{L^2} \simeq \|f\|_{\dot{H}^1}, \quad \forall f \in C^\infty(\mathbb{R}^3).$$

For real-valued  $u = (u_1, u_2) \in \mathcal{H} = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ , we write

$$(5.13) \quad U := Au_1 + iu_2.$$

From (5.12), we know

$$(5.14) \quad \|U\|_{L^2} \simeq \|(u_1, u_2)\|_{\mathcal{H}}.$$

We also notice that  $u$  solves (5.8) if and only if

$$(5.15) \quad U := Au + i\partial_t u$$

satisfies

$$(5.16) \quad i\partial_t U = AU + V_1 u + V_2(x - vt)u,$$

$$(5.17) \quad U(0) = Ag + if \in L^2(\mathbb{R}^3).$$

By Duhamel's formula,

$$(5.18) \quad U(t) = e^{itA}U(0) - i \int_0^t e^{-i(t-s)A} (V_1 u(s) + V_2(\cdot - vs)u(s)) ds.$$

Let  $P := A^{-1}\Re$ , then from Strichartz estimates for the free evolution,

$$(5.19) \quad \|Pe^{itA}U(0)\|_{L_t^p L_x^q} \lesssim \|U(0)\|_{L^2}.$$

Writing  $V_1 = V_3 V_4$ ,  $V_2 = V_5 V_6$ , since  $V_1$  and  $V_2$  decay like  $\langle x \rangle^{-\alpha}$  with  $\alpha > 3$ , we can make  $V_3$  and  $V_5$  satisfy the weight condition in Theorem 2.5. Also  $V_4^2$ ,  $V_6^2$  decay with rate  $\langle x \rangle^{-\alpha}$ . By the Christ-Kiselev lemma, cf. Lemma 2.6, it suffices to bound

$$(5.20) \quad \left\| P \int_0^\infty e^{-i(t-s)A} V_3 V_4 u(s) ds \right\|_{L_t^p L_x^q},$$

and

$$(5.21) \quad \left\| P \int_0^\infty e^{-i(t-s)A} V_5 V_6 (\cdot - vs) u(s) ds \right\|_{L_t^p L_x^q}.$$

It is clear that

$$(5.22) \quad \left\| P \int_0^\infty e^{-i(t-s)A} V_3 V_4 u(s) ds \right\|_{L_t^p L_x^q} \leq \|K\|_{L_{t,x}^2 \rightarrow L_t^p L_x^q} \|V_4 u\|_{L_{t,x}^2},$$

where

$$(5.23) \quad (KF)(t) := P \int_0^\infty e^{-i(t-s)A} V_3 F(s) ds.$$

Similarly,

$$(5.24) \quad \left\| P \int_0^\infty e^{-i(t-s)A} V_5 V_6 (\cdot - vs) u(s) ds \right\|_{L_t^p L_x^q} \leq \|\tilde{K}\|_{L_{t,x}^2 \rightarrow L_t^p L_x^q} \|V_6(x - vt)u\|_{L_{t,x}^2},$$

where

$$(5.25) \quad (\tilde{K}F)(t) := P \int_0^\infty e^{-i(t-s)A} V_5 (\cdot - vs) F(s) ds.$$

We need to estimate

$$(5.26) \quad \|K\|_{L_{t,x}^2 \rightarrow L_t^p L_x^q}, \|\tilde{K}\|_{L_{t,x}^2 \rightarrow L_t^p L_x^q}.$$

Testing against  $F \in L_{t,x}^2$ , clearly,

$$(5.27) \quad \|KF\|_{L_t^p L_x^q} \leq \|Pe^{-itA}\|_{L^2 \rightarrow L_t^p L_x^q} \left\| \int_0^\infty e^{isA} V_3 F(s) ds \right\|_{L^2}.$$

$$(5.28) \quad \|\tilde{K}F\|_{L_t^p L_x^q} \leq \|Pe^{-itA}\|_{L^2 \rightarrow L_t^p L_x^q} \left\| \int_0^\infty e^{isA} V_5 (\cdot - vs) F(s) ds \right\|_{L^2}.$$

The first factors on the right-hand side of (5.27) and (5.28) is bounded by Strichartz estimates for the free evolution. Consider the second factors, by duality, it suffices to show

$$(5.29) \quad \|V_3 e^{-itA} \phi\|_{L_{t,x}^2} \lesssim \|\phi\|_{L^2}, \forall \phi \in L^2(\mathbb{R}^3)$$

$$(5.30) \quad \|V_5(x - vt) e^{-itA} \phi\|_{L_{t,x}^2} \lesssim \|\phi\|_{L^2}, \forall \phi \in L^2(\mathbb{R}^3).$$

which holds by Theorem 2.5 and Corollary 2.10.

Hence

$$(5.31) \quad \left\| \int_0^\infty e^{isA} V_3 F(s) ds \right\|_{L^2} \lesssim \|F\|_{L_{t,x}^2},$$

$$(5.32) \quad \left\| \int_0^\infty e^{isA} V_5 (\cdot - vs) F(s) ds \right\|_{L^2} \lesssim \|F\|_{L_{t,x}^2}.$$

Therefore, indeed, we have

$$(5.33) \quad \|K\|_{L_{t,x}^2 \rightarrow L_t^p L_x^q} \leq C, \|\tilde{K}\|_{L_{t,x}^2 \rightarrow L_t^p L_x^q} \leq C$$

and from (5.22), it follows that

$$(5.34) \quad \left\| P \int_0^\infty e^{-i(t-s)A} V_3 V_4 u(s) ds \right\|_{L_t^p L_x^q} \lesssim \|V_4 u\|_{L_{t,x}^2},$$

$$(5.35) \quad \left\| P \int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs) u(s) ds \right\|_{L_t^p L_x^q} \lesssim \|V_6(x - vt)u\|_{L_{t,x}^2}.$$

By estimates (5.4) and (5.5) from Corollary 5.1,

$$(5.36) \quad \|V_4 u\|_{L_{t,x}^2} \lesssim \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^\alpha} |u(x, t)|^2 dx dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1},$$

$$(5.37) \quad \|V_6(x - vt)u\|_{L_{t,x}^2} \lesssim \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x - vt \rangle^\alpha} |u(x, t)|^2 dx dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

They follows that

$$(5.38) \quad \left\| P \int_0^\infty e^{-i(t-s)A} V_3 V_4 u(s) ds \right\|_{L_t^p L_x^q} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

$$(5.39) \quad \left\| P \int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs) u(s) ds \right\|_{L_t^p L_x^q} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

Therefore, by estimates (5.19), (5.38) and (5.39), for  $p > 2$ , and

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}$$

we have

$$(5.40) \quad \|u\|_{L_t^p([0, \infty), L_x^q)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

as claimed □

Taking the case  $p = q$  in the regular Strichartz estimate (5.11) and interpolating it with the endpoint reversed Strichartz estimate (4.6), we obtain more reversed Strichartz estimates.

**Corollary 5.3.** *Let  $|v| < 1$ . Suppose  $u$  is a scattering state in the sense of Definition 1.2 which solves*

$$(5.41) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

*with initial data*

$$(5.42) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then for  $(p, q)$  satisfying*

$$(5.43) \quad \frac{1}{2} = \frac{1}{p} + \frac{3}{q}$$

*with*

$$(5.44) \quad 2 \leq p \leq 8,$$

*we have*

$$(5.45) \quad \|u\|_{L_x^q(\mathbb{R}^3, L_t^p[0, \infty))} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

**Theorem 5.4.** *Let  $|v| < 1$ . Suppose  $u$  is a scattering state in the sense of Definition 1.2 which solves*

$$(5.46) \quad \partial_{tt}u - \Delta u + V_1(x)u + V_2(x-vt)u = 0$$

*with initial data*

$$(5.47) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then we have*

$$(5.48) \quad \sup_{t \geq 0} (\|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}) \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

*Proof.* The proof is similar to Theorem 5.2. We still use the notations from the above proof of Theorem 5.2. Set

$$U := Au + i\partial_t u,$$

then by Duhamel's formula,

$$(5.49) \quad U(t) = e^{itA}U(0) - i \int_0^t e^{-i(t-s)A} (V_1 u(s) + V_2(\cdot - vs)u(s)) ds.$$

It suffices to estimate the  $L^2$  norm of  $U(t)$ .

From the energy estimate for the free evolution,

$$(5.50) \quad \sup_{t \geq 0} \|e^{itA}U(0)\|_{L_x^2} \lesssim \|U(0)\|_{L^2}.$$

Writing  $V_1 = V_3V_4$ ,  $V_2 = V_5V_6$  as above, it suffices to bound

$$(5.51) \quad \sup_{t \geq 0} \left\| \int_0^\infty e^{-i(t-s)A} V_3 V_4 u(s) ds \right\|_{L_x^2},$$

and

$$(5.52) \quad \sup_{t \geq 0} \left\| \int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs)u(s) ds \right\|_{L_x^2}.$$

It is clear that

$$(5.53) \quad \sup_{t \geq 0} \left\| \int_0^\infty e^{-i(t-s)A} V_3 V_4 u(s) ds \right\|_{L_x^2} \leq \|K\|_{L_t^2 L_x^2 \rightarrow L_t^\infty L_x^2} \|V_4 u\|_{L_{t,x}^2},$$

where

$$(5.54) \quad (KF)(t) := \int_0^\infty e^{-i(t-s)A} V_3 F(s) ds.$$

Similarly,

$$(5.55) \quad \left\| \int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs)u(s) ds \right\|_{L_t^\infty L_x^2} \leq \|\tilde{K}\|_{L_t^2 L_x^2 \rightarrow L_t^\infty L_x^2} \|V_6(x-vt)u\|_{L_t^2 L_x^2},$$

where

$$(5.56) \quad (\tilde{K}F)(t) := \int_0^\infty e^{-i(t-s)A} V_3(\cdot - vs)F(s) ds.$$

We need to estimate

$$(5.57) \quad \|K\|_{L_t^2 L_x^2 \rightarrow L_t^\infty L_x^2}, \quad \|\tilde{K}\|_{L_t^2 L_x^2 \rightarrow L_t^\infty L_x^2}.$$

Testing against  $F \in L_t^2([0, \infty), L_x^2)$ , clearly,

$$(5.58) \quad \|KF\|_{L_t^\infty L_x^2} \leq \|e^{-itA}\|_{L^2 \rightarrow L_t^\infty L_x^2} \left\| \int_0^\infty e^{isA} V_3 F(s) ds \right\|_{L^2}.$$

$$(5.59) \quad \left\| \tilde{K}F \right\|_{L_t^\infty L_x^2} \leq \left\| e^{-itA} \right\|_{L^2 \rightarrow L_t^\infty L_x^2} \left\| \int_0^\infty e^{isA} V_5(\cdot - vs) F(s) ds \right\|_{L^2}.$$

The first factors on the right-hand side of (5.58) and (5.59) is bounded by the energy estimates for the free evolution. The second factors are estimated in the same manner as for (5.27) and (5.28).

Therefore, we have

$$(5.60) \quad \|K\|_{L_t^2 L_x^2 \rightarrow L_t^\infty L_x^2} \leq C, \quad \left\| \tilde{K} \right\|_{L_t^2 L_x^2 \rightarrow L_t^\infty L_x^2} \leq C$$

and from (5.53),

$$(5.61) \quad \sup_{t \geq 0} \left\| \int_0^\infty e^{-i(t-s)A} V_3 V_4 u(s) ds \right\|_{L_x^2} \lesssim \|V_4 u\|_{L_{t,x}^2},$$

$$(5.62) \quad \sup_{t \geq 0} \left\| \int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs) u(s) ds \right\|_{L_x^2} \lesssim \|V_6(x - vt)u\|_{L_{t,x}^2}.$$

From Corollary 5.1,

$$(5.63) \quad \|V_4 u\|_{L_t^2 L_x^2} \lesssim \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^\alpha} |u(x, t)|^2 dx dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1},$$

$$(5.64) \quad \|V_6(x - vt)u\|_{L_t^2 L_x^2} \lesssim \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x - vt \rangle^\alpha} |u(x, t)|^2 dx dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

They imply

$$(5.65) \quad \sup_{t \geq 0} \left\| \int_0^\infty e^{-i(t-s)A} V_3 V_4 u(s) ds \right\|_{L_x^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

$$(5.66) \quad \sup_{t \geq 0} \left\| \int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs) u(s) ds \right\|_{L_x^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

Therefore, with estimates (5.50), (5.65) and (5.66), we have

$$(5.67) \quad \sup_{t \geq 0} (\|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}) \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

as claimed □

Similarly, one can also obtain the local energy decay estimate:

**Theorem 5.5.** *Let  $|v| < 1$ . Suppose  $u$  is a scattering state in the sense of Definition 1.2 which solves*

$$(5.68) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

*with initial data*

$$(5.69) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then for  $\forall \epsilon > 0$ ,  $|\mu| < 1$ , we have*

$$(5.70) \quad \left\| (1 + |x - \mu t|)^{-\frac{1}{2}-\epsilon} (|\nabla u| + |u_t|) \right\|_{L^2([0, \infty), L_x^2)} \lesssim_{\mu, \epsilon} \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

*Proof.* The proof is the same as above with the energy estimate for the free wave equation replaced by the local energy decay estimate for the free wave equation.

$$(5.71) \quad \left\| (1 + |x - \mu t|)^{-\frac{1}{2}-\epsilon} e^{it\sqrt{-\Delta}} f \right\|_{L^2([0, \infty), L_x^2)} \lesssim_{\mu, \epsilon} \|f\|_{L_x^2}.$$

The claim follows easily. □

Finally, we consider the boundedness of the total energy. We denote the total energy by

$$(5.72) \quad E(t) = \int |\nabla_x u|^2 + |\partial_t u|^2 + V_1 |u|^2 + V_2(x - vt) |u|^2 dx.$$

**Corollary 5.6.** *Let  $|v| < 1$  and suppose  $u$  is a scattering state in the sense of Definition 1.2 and solves*

$$(5.73) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

*with initial data*

$$(5.74) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Assume*

$$(5.75) \quad \|\nabla V_2\|_{L^1} < \infty,$$

*then  $E(t)$  is bounded by the initial energy independently of  $t$ ,*

$$\sup_{t \geq 0} E(t) \lesssim \|(g, f)\|_{\dot{H}^1 \times L^2}^2.$$

*Proof.* We might assume  $u$  is smooth. Taking time derivative of  $E(t)$ , with the fact  $u$  solves

$$(5.76) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x - vt)u = 0,$$

one obtains

$$(5.77) \quad \partial_t E(t) = \int_{\mathbb{R}^3} \partial_t V_2(x - vt) |u|^2(x, t) dx = -v \int_{\mathbb{R}^3} \partial_x V_2(x) |u^S(x, t)|^2 dx$$

by a simple change of variable.

Note that

$$(5.78) \quad \begin{aligned} \int_0^\infty |\partial_t E(t)| dt &\lesssim \int_0^\infty \int_{\mathbb{R}^3} |\partial_x V_2(x)| |u^S(x, t)|^2 dx dt \\ &= \|\partial_x V_2\|_{L_x^1} \|u^S\|_{L_x^\infty L_t^2[0, \infty)}^2 \\ &\lesssim \|(g, f)\|_{\dot{H}^1 \times L^2}^2. \end{aligned}$$

For arbitrary  $t \in \mathbb{R}$ , we have

$$(5.79) \quad E(t) - E(0) \leq \int_{\mathbb{R}^+} |\partial_t E(t)| dt \lesssim \|(g, f)\|_{\dot{H}^1 \times L^2}^2$$

which implies

$$(5.80) \quad \sup_{t \geq 0} E(t) \lesssim \|(g, f)\|_{\dot{H}^1 \times L^2}^2$$

as claimed. □

## 6. INHOMOGENEOUS ESTIMATES

When we consider nonlinear applications, it is useful to have estimates for inhomogeneous equations. Again, for simplicity we consider the case of two potentials.

**6.1. Scattering states.** We start with revisiting scattering states.

Recall that if  $u$  solves

$$(6.1) \quad \partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

and  $u$  satisfies

$$(6.2) \quad \|P_b(H_1)u(t)\|_{L_x^2} \rightarrow 0, \quad \|P_b(H_2)u_L(t')\|_{L_{x'}^2} \rightarrow 0, \quad t, t' \rightarrow \infty,$$

then we call it a scattering state.

Clearly, the set of  $(g, f) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$  which produce a scattering state forms a subspace of  $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ . In order to study this more precisely, we reformulate the wave equation as a Hamiltonian system,

$$(6.3) \quad \partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta + V_1(x) + V_2(x - vt) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = 0.$$

Setting

$$(6.4) \quad U := \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$(6.5) \quad H(t) := \begin{pmatrix} -\Delta + V_1(x) + V_2(x - vt) & 0 \\ 0 & 1 \end{pmatrix},$$

and defining

$$(6.6) \quad P_1(U) := u,$$

we can rewrite the wave equation with charge transfer Hamiltonian as

$$(6.7) \quad \dot{U} - JH(t)U = 0,$$

$$(6.8) \quad U(0) = \begin{pmatrix} g \\ f \end{pmatrix}.$$

With the above notations, we define the solution operator starting from  $\tau$  to  $t$  as  $S(t, \tau)$ . In particular, one can write

$$(6.9) \quad U(t) = S(t, 0)U(0).$$

As pointed out above, the set of  $(g, f) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$  which produce a scattering state in the sense of Definition 1.2 forms a subspace

$$\mathcal{H}_s(0) \subset H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3).$$

We can do a more general time-dependent construction. One considers the evolution from  $\tau$  to  $t$ , i.e.,  $S(t, \tau)$ . Similar as our original construction there is a subspace

$$\mathcal{H}_s(\tau) \subset H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$$

such that for  $\Phi \in \mathcal{H}_s(\tau)$ ,

$$(6.10) \quad \|P_b(H_1)S(t, \tau)\Phi\|_{L_x^2} \rightarrow 0, \quad \|P_b(H_2)(S(\cdot, \tau)\Phi)_{L_\tau}(t')\|_{L_{x'}^2} \rightarrow 0, \quad t, t' \rightarrow \infty.$$

It is important to notice a fundamental property of  $\mathcal{H}_s(\tau)$ .

**Lemma 6.1.** *Denote  $P_s(\tau)$  as the projection onto  $\mathcal{H}_s(\tau)$ . Then  $\forall s, \tau \in \mathbb{R}$ ,*

$$(6.11) \quad P_s(s)S(s, \tau) = S(s, \tau)P_s(\tau).$$



*Proof.* Notice that for  $\Phi \in \mathcal{H}_s(\tau)$ , then  $S(s, \tau)\Phi \in \mathcal{H}_s(s)$ . Since

$$(6.12) \quad \|P_b(H_1)S(t, s)S(s, \tau)\Phi\|_{L^2} = \|P_b(H_1)S(t, \tau)\Phi\|_{L^2} \rightarrow 0,$$

$$(6.13) \quad \|P_b(H_2)(S(\cdot, s)S(s, \tau)\Phi)_{L_s}(t')\|_{L_{x'}^2} = \|P_b(H_2)(S(\cdot, \tau)\Phi)_{L_\tau}(t')\|_{L_{x'}^2} \rightarrow 0$$

as  $t, t' \rightarrow \infty$  by the definition of  $\mathcal{H}_s(\tau)$ . Then again by the definition of  $\mathcal{H}_s(s)$ , it is clear  $S(s, \tau)\Phi \in \mathcal{H}_s(s)$ . Conversely, by symmetry, for  $\Phi \in \mathcal{H}_s(s)$ , then  $S(\tau, s)\Phi \in \mathcal{H}_s(\tau)$ . Therefore, we have that the scattering spaces are invariant under the flow  $S(s, \tau)$ ,

$$(6.14) \quad \mathcal{H}_s(s) = S(s, \tau)\mathcal{H}_s(\tau).$$

Let  $\Phi \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ , then  $S(s, \tau)P_s(\tau)\Phi \in \mathcal{H}_s(s)$  by construction. So

$$(6.15) \quad \begin{aligned} S(s, \tau)P_s(\tau)\Phi &= (1 - P_s(s))S(s, \tau)P_s(\tau)\Phi + P_s(s)S(s, \tau)P_s(\tau)\Phi \\ &= P_s(s)S(s, \tau)P_s(\tau)\Phi. \end{aligned}$$

Similarly,

$$(6.16) \quad P_s(s)S(s, \tau)\Phi = P_s(s)S(s, \tau)P_s(\tau)\Phi.$$

Hence

$$(6.17) \quad P_s(s)S(s, \tau) = S(s, \tau)P_s(\tau),$$

as claimed.  $\square$

For wave equations, it is always necessary to exchange the scalar formulation and the Hamiltonian formulation. Here we introduce some notations which are useful in our later analysis. We define  $P_s(\tau)$  via the Hamiltonian formulation above. Now consider a scalar function  $v(x, t) \in C(\mathbb{R}, H^1(\mathbb{R}^3)) \cap C(\mathbb{R}, L^2(\mathbb{R}^3))$ , it can give the data  $(v, v_t)$  for the charge transfer model. We define

$$(6.18) \quad P_s^{\mathbf{S}}(\tau)v := P_1P_s(\tau)(v, v_t),$$

where  $P_1$  is the projection onto the first component as in (6.6). For a vector-valued function  $V = \begin{pmatrix} v \\ v_t \end{pmatrix} \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ ,

$$(6.19) \quad P_s^{\mathbf{V}}(\tau)V := P_1P_s(\tau)V,$$

Given data  $(g, f) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ , formally, we can define the evolution from  $\tau$  to  $t$  associated with  $f$  as

$$(6.20) \quad U(t, \tau)f$$

and the evolution associated with  $g$  as

$$(6.21) \quad \dot{U}(t, \tau)g.$$

Here  $\dot{U}$  is just a formal notation.

Finally, we consider two special cases.

Setting  $g = 0$ , then the set of  $f \in L^2(\mathbb{R}^3)$  such that  $(0, f) \in \mathcal{H}_s(\tau)$  forms a subspace of  $L^2(\mathbb{R}^3)$ . We use  $L_s^L(\tau)$  to denote this subspace and let  $P_s^L(\tau)$  to be the associated projection.

Setting  $f = 0$ , then the set of  $g \in H^1(\mathbb{R}^3)$  such that  $(g, 0) \in \mathcal{H}_s(\tau)$  forms a subspace of  $H^1(\mathbb{R}^3)$ . We use  $H_s^1(\tau)$  to denote this subspace and let  $P_s^H(\tau)$  to be the associated projection.

**6.2. Inhomogeneous local decay estimate and Strichartz estimates.** Throughout this subsection, we will use the short-hand notation

$$(6.22) \quad L_t^p L_x^q := L_t^p([0, \infty), L_x^q).$$

Let  $u$  solve

$$(6.23) \quad \partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

with initial data

$$(6.24) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

Denote the evolution as

$$(6.25) \quad u(x, t) = U(t, 0)f + \dot{U}(t, 0)g.$$

From the endpoint reversed Strichartz estimate, Theorem 4.1, with the notations introduced above, we know

$$(6.26) \quad \sup_x \int_0^\infty |P_s^{\mathbf{S}}(t) u(x, t)|^2 dt \lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2$$

and

$$(6.27) \quad \sup_x \int_0^\infty |(P_s^{\mathbf{S}}(t) u(x, t))^S|^2 dt \lesssim (\|f\|_{L^2} + \|g\|_{\dot{H}^1})^2.$$

Furthermore, we have the following corollary as particular situations:

**Corollary 6.2.** *For the evolution  $U(t, \tau)$  and the projections  $P_s^{\mathbf{S}}(t)$ ,  $P_s^L(\tau)$  defined above, one has*

$$(6.28) \quad \sup_x \int_0^\infty |P_s^{\mathbf{S}}(t) U(t, \tau)f|^2 dt = \sup_x \int_0^\infty |U(t, \tau)P_s^L(\tau)f|^2 dt \lesssim \|f\|_{L^2}^2,$$

$$(6.29) \quad \sup_x \int_0^\infty |(P_s^{\mathbf{S}}(t) U(t, \tau)f)^S|^2 dt = \sup_x \int_0^\infty |U^S(t, \tau)P_s^L(\tau)f|^2 dt \lesssim \|f\|_{L^2}^2,$$

where  $U^S$  denotes the integration along the slanted line  $(x + vt, t)$ .

*Proof.* This is just the particular cases of what we have discussed above. □

By Corollary 6.2, we have the weighted estimates for the inhomogeneous evolution.

**Lemma 6.3.** *For  $\alpha > 3$ , with  $U(t, \tau)$  and projections  $P_s^{\mathbf{S}}(t)$ ,  $P_s^L(\tau)$  defined above, we have*

$$(6.30) \quad \left\| \langle x \rangle^{-\frac{\alpha}{2}} \int_0^t U(t, \tau) P_s^L(\tau) H(\tau) d\tau \right\|_{L_t^2 L_x^2} = \left\| \langle x \rangle^{-\frac{\alpha}{2}} \int_0^t P_s^{\mathbf{S}}(t) U(t, \tau) H(\tau) d\tau \right\|_{L_t^2 L_x^2},$$

$$(6.31) \quad \left\| \langle x \rangle^{-\frac{\alpha}{2}} \int_0^t U(t, \tau) P_s^L(\tau) H(\tau) d\tau \right\|_{L_t^2 L_x^2} \lesssim \|H(t)\|_{L_t^1 L_x^2},$$

$$(6.32) \quad \left\| \langle x \rangle^{-\frac{\alpha}{2}} \int_0^t U^S(t, \tau) P_s^L(\tau) H(\tau) d\tau \right\|_{L_t^2 L_x^2} = \left\| \langle x \rangle^{-\frac{\alpha}{2}} \int_0^t (P_s^{\mathbf{S}}(t) U(t, \tau))^S H(\tau) d\tau \right\|_{L_t^2 L_x^2},$$

$$(6.33) \quad \left\| \langle x - vt \rangle^{-\frac{\alpha}{2}} \int_0^t U(t, \tau) P_s^L(\tau) H(\tau) d\tau \right\|_{L_t^2 L_x^2} \lesssim \|H(t)\|_{L_t^1 L_x^2}.$$

*Proof.* By the definition of projections, we have

$$\begin{aligned} \left\| \langle x \rangle^{-\frac{\alpha}{2}} \int_0^t U(t, \tau) P_s^L(\tau) H(\tau) d\tau \right\|_{L_t^2 L_x^2} &= \left\| \langle x \rangle^{-\frac{\alpha}{2}} \int_0^t P_s^S(t) U(t, \tau) H(\tau) d\tau \right\|_{L_t^2 L_x^2}, \\ \left\| \langle x \rangle^{-\frac{\alpha}{2}} \int_0^t U^S(t, \tau) P_s^L(\tau) H(\tau) d\tau \right\|_{L_t^2 L_x^2} &= \left\| \langle x \rangle^{-\frac{\alpha}{2}} \int_0^t (P_s^S(t) U(t, \tau))^S H(\tau) d\tau \right\|_{L_t^2 L_x^2}. \end{aligned}$$

Applying Minkowski's inequality and Corollary 6.2, we have

$$\begin{aligned} \left\| \langle x \rangle^{-\alpha} \int_0^t U(t, \tau) P_s^L(\tau) H(\tau) d\tau \right\|_{L_t^2 L_x^2} &\lesssim \left\| \langle x \rangle^{-\alpha} \int_0^t |U(t, \tau) P_s^L(\tau) H(\tau)| d\tau \right\|_{L_t^2 L_x^2} \\ &\lesssim \left\| \langle x \rangle^{-\alpha} \int_0^\infty |U(t, \tau) P_s^L(\tau) H(\tau)| d\tau \right\|_{L_t^2 L_x^2} \\ &\lesssim \int_0^\infty \|U(t, \tau) P_s^L(\tau) H(\tau)\|_{L_x^\infty L_t^2} d\tau \\ &\lesssim \|H(t)\|_{L_t^1 L_x^2} \end{aligned}$$

and similarly,

$$\begin{aligned} \left\| \langle x - vt \rangle^{-\alpha} \int_0^t U(t, \tau) P_s^L(\tau) H(\tau) d\tau \right\|_{L_t^2 L_x^2} &\lesssim \int_0^\infty \|U^S(t, \tau) P_s^L(\tau) H(\tau)\|_{L_x^\infty L_t^2} d\tau \\ &\lesssim \|H(t)\|_{L_t^1 L_x^2}. \end{aligned}$$

The lemma is proved.  $\square$

With the preparations above, we are ready to proceed to the analysis of inhomogeneous Strichartz estimates. As one can observe from previous sections on the homogeneous Strichartz estimates that it suffices to establish certain local decay estimates.

Now we set

$$(6.34) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x - vt)u = F$$

with initial data

$$(6.35) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x)$$

**Lemma 6.4.** *Suppose  $u$  solves*

$$(6.36) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x - vt)u = F$$

*with initial data*

$$(6.37) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then for  $\alpha > 3$   $|v| < 1$ , we have*

$$(6.38) \quad \left\| \langle x \rangle^{-\frac{\alpha}{2}} P_s^S(t) u \right\|_{L_t^2([0, \infty), L_x^2)} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} + \|F\|_{L_t^1([0, \infty), L_x^2)},$$

*and*

$$(6.39) \quad \left\| \langle x - vt \rangle^{-\frac{\alpha}{2}} P_s^S(t) u \right\|_{L_t^2([0, \infty), L_x^2)} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} + \|F\|_{L_t^1([0, \infty), L_x^2)}.$$

*Proof.* By Duhamel's formula, we write

$$(6.40) \quad u(x, t) = U(t, 0)f + \dot{U}(t, 0)g + \int_0^t U(t, s)F(s) ds.$$

$$\begin{aligned}
P_s^{\mathbf{S}}(t)u(x, t) &= P_s^{\mathbf{S}}(t) \left( U(t, 0)f + \dot{U}(t, 0)g \right) + \int_0^t P_s^{\mathbf{S}}(t) U(t, s)F(s) ds \\
(6.41) \quad &= P_s^{\mathbf{S}}(t) \left( U(t, 0)f + \dot{U}(t, 0)g \right) + \int_0^t U(t, s)P_s^L(s)F(s) ds.
\end{aligned}$$

Applying the weighted norms, for the homogeneous part, we know

$$(6.42) \quad \left\| \langle x \rangle^{-\alpha} P_s^{\mathbf{S}}(t) \left( U(t, 0)f + \dot{U}(t, 0)g \right) \right\|_{L_t^2 L_x^2} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2}$$

and

$$(6.43) \quad \left\| \langle x - vt \rangle^{-\alpha} P_s^{\mathbf{S}}(t) \left( U(t, 0)f + \dot{U}(t, 0)g \right) \right\|_{L_t^2 L_x^2} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2}.$$

For the inhomogeneous part, by our discussion above, one has

$$(6.44) \quad \left\| \langle x \rangle^{-\alpha} \int_0^t U(t, s)P_s^L(s)F(s) ds \right\|_{L_t^2 L_x^2} \lesssim \|F\|_{L_t^1 L_x^2},$$

and

$$(6.45) \quad \left\| \langle x - vt \rangle^{-\alpha} \int_0^t U(t, s)P_s^L(s)F(s) ds \right\|_{L_t^2 L_x^2} \lesssim \|F\|_{L_t^1 L_x^2}.$$

Therefore, one can conclude that

$$(6.46) \quad \left\| \langle x \rangle^{-\frac{\alpha}{2}} P_s^{\mathbf{S}}(t)u \right\|_{L_t^2([0, \infty), L_x^2)} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} + \|F\|_{L_t^1([0, \infty), L_x^2)},$$

$$(6.47) \quad \left\| \langle x - vt \rangle^{-\frac{\alpha}{2}} P_s^{\mathbf{S}}(t)u \right\|_{L_t^2([0, \infty), L_x^2)} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} + \|F\|_{L_t^1([0, \infty), L_x^2)}.$$

The lemma is proved.  $\square$

With the decay estimate Lemma 6.4, we can establish Strichartz estimates using almost identical procedures as for the homogeneous Strichartz estimates.

**Theorem 6.5.** *Let  $|v| < 1$  and suppose  $u$  solves*

$$(6.48) \quad \partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = F$$

*with initial data*

$$(6.49) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then for  $p, \tilde{p} > 2$ , and*

$$(6.50) \quad \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \quad \frac{1}{2} = \frac{1}{\tilde{p}} + \frac{3}{\tilde{q}}$$

*we have*

$$(6.51) \quad \|P_s^{\mathbf{S}}(t)u\|_{L_t^p([0, \infty), L_x^q)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_t^{\tilde{p}'}([0, \infty), L_x^{\tilde{q}'}) \cap L_t^1([0, \infty), L_x^2)}$$

*where  $\tilde{p}', \tilde{q}'$  are Hölder conjugate of  $\tilde{p}, \tilde{q}$ .*

*Proof.* The proof is almost identical to the one for Theorem 5.2. But we need some preliminary calculations. By Lemma 6.1, we know

$$(6.52) \quad P_s(s)S(s, \tau) = S(s, \tau)P_s(\tau).$$

Differentiating (6.52) with respect to  $s$  and then setting both  $\tau = s = t$ , we have

$$(6.53) \quad \dot{P}_s(t) = -JH(t)P_s(t) + P_s(t)JH(t).$$

Just as we discussed about projections, we write

$$(6.54) \quad \partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = F$$

as a system:

$$(6.55) \quad \partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta + V_1(x) + V_2(x-vt) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}.$$

Then

$$(6.56) \quad P_s(t) \partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + P_s(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta + V_1(x) + V_2(x-vt) & 0 \\ 0 & 1 \end{pmatrix} = P_s(t) \begin{pmatrix} 0 \\ F(t) \end{pmatrix}$$

which is

$$(6.57) \quad P_s(t) \dot{U}(t) - P_s(t) JH(t) = P_s(t) F(t).$$

By equations (6.52) and (6.57), one has

$$(6.58) \quad \frac{d}{dt} (P_s(t) U(t)) - JH(t) P_s(t) U(t) = P_s(t) F(t).$$

Hence returning to our scalar setting, we have

$$(6.59) \quad \partial_{tt} (P_s^{\mathbf{S}}(t) u) + (-\Delta + V_1(x) + V_2(x-vt)) P_s^{\mathbf{S}}(t) u = P_s^{\mathbf{S}}(t) F(t).$$

Now we are ready to proceed to the Strichartz estimates argument similar to the case in Theorem 5.2.

Again, following [LSch], setting  $A = \sqrt{-\Delta}$  and taking

$$U(t) = A P_s^{\mathbf{S}}(t) u(t) + i \partial_t (P_s^{\mathbf{S}}(t) u(t)),$$

then  $U$  satisfies

$$(6.60) \quad i \partial_t U = AU + V_1 P_s^{\mathbf{S}}(t) u(t) + V_2(x-vt) P_s^{\mathbf{S}}(t) u(t) + P_s^{\mathbf{S}}(t) F,$$

By Duhamel's formula,

$$(6.61) \quad U(t) = e^{itA} U(0) - i \int_0^t e^{-i(t-s)A} (V_1 P_s^{\mathbf{S}}(s) u(s) + V_2(\cdot - vs) P_s^{\mathbf{S}}(s) u(s) + P_s^{\mathbf{S}}(s) F(s)) ds.$$

Let  $P := A^{-1} \Re$ , then from Strichartz estimates for the free evolution,

$$(6.62) \quad \|P e^{itA} U(0)\| \lesssim \|U(0)\|_{L^2},$$

and

$$(6.63) \quad \left\| \int_0^t e^{-i(t-s)A} P_s^{\mathbf{S}}(s) F(s) ds \right\|_{L_t^p L_x^q} \lesssim \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}.$$

As in the proof of Theorem 5.2, writing  $V_1 = V_3 V_4$ ,  $V_2 = V_5 V_6$ , it suffices to bound

$$(6.64) \quad \left\| P \int_0^\infty e^{-i(t-s)A} V_3 V_4 P_s^{\mathbf{S}}(s) u(s) ds \right\|_{L_t^p L_x^q},$$

and

$$(6.65) \quad \left\| P \int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs) P_s^{\mathbf{S}}(s) u(s) ds \right\|_{L_t^p L_x^q}.$$

In the same manner as we did in the proof of Theorem 5.2, one has

$$(6.66) \quad \left\| P \int_0^\infty e^{-i(t-s)A} V_3 V_4 P_s^{\mathbf{S}}(s) u(s) ds \right\|_{L_t^p L_x^q} \lesssim \|V_4 P_s^{\mathbf{S}}(t) u\|_{L_t^2 L_x^2},$$

$$(6.67) \quad \left\| P \int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs) P_s^{\mathbf{S}}(s) u(s) ds \right\|_{L_t^p L_x^q} \lesssim \|V_6(x-vt) P_s^{\mathbf{S}}(t) u\|_{L_t^2 L_x^2}.$$

By estimates (6.38) and (6.39) from Lemma 6.4,

$$(6.68) \quad \|V_4 P_s^{\mathbf{S}}(t) u\|_{L_t^2 L_x^2} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} + \|F\|_{L_t^1 L_x^2},$$

$$(6.69) \quad \|V_6(x - vt)P_s^{\mathbf{S}}(t)u\|_{L_t^2 L_x^2} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} + \|F\|_{L_t^1 L_x^2}.$$

Therefore, by the same argument as for the homogeneous Strichartz estimates, we have

$$(6.70) \quad \|P_s^{\mathbf{S}}(t)u\|_{L_t^p([0, \infty), L_x^q)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_t^{\tilde{p}'}([0, \infty), L_x^{\tilde{q}'}) \cap L_t^1([0, \infty), L_x^2)}.$$

as claimed  $\square$

From the discussions above, we can also conclude the endpoint reversed Strichartz estimate.

**Theorem 6.6.** *Let  $|v| < 1$  and suppose  $u$  solves*

$$(6.71) \quad \partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = F$$

*with initial data*

$$(6.72) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then we have*

$$(6.73) \quad \sup_x \int_0^\infty |P_s^{\mathbf{S}}(t)u|^2 dt \lesssim \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_t^1([0, \infty), L_x^2)} \right)^2$$

Taking the case  $p = q$  in the regular Strichartz estimate and interpolating it with the endpoint reversed Strichartz estimate (6.73), we obtain more reversed Strichartz estimates.

**Corollary 6.7.** *Let  $|v| < 1$  and suppose  $u$  solves*

$$(6.74) \quad \partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = F$$

*with initial data*

$$u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

*Then for*

$$(6.75) \quad 2 \leq p, \tilde{p} \leq 8$$

*and*

$$(6.76) \quad \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \quad \frac{1}{2} = \frac{1}{\tilde{p}} + \frac{3}{\tilde{q}}$$

*we have*

$$(6.77) \quad \|P_s^{\mathbf{S}}(t)u\|_{L_x^q(\mathbb{R}^3, L_t^p[0, \infty))} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\tilde{q}'}(\mathbb{R}^3, L_t^{\tilde{p}'}[0, \infty)) \cap L_t^1([0, \infty), L_x^2)}.$$

*where  $\tilde{p}'$ ,  $\tilde{q}'$  are Hölder conjugate of  $\tilde{p}$ ,  $\tilde{q}$ .*

## 7. SCATTERING

In this section, we show some applications of the results in this paper. We will study the long-time behaviors for a scattering state in the sense of Definition 1.2.

Following the notations from above section, we will still use the short-hand notation

$$(7.1) \quad L_t^p L_x^q := L_t^p([0, \infty), L_x^q).$$

In general, we can write a general wave equation as

$$(7.2) \quad \partial_{tt}u - \Delta u = F(u, t)$$

with initial data

$$(7.3) \quad u(x, 0) = g(x), \quad u_t(x, 0) = f(x).$$

Also consider the homogeneous free wave equation,

$$(7.4) \quad \partial_{tt}u_0 - \Delta u_0 = 0$$

with initial data

$$(7.5) \quad u_0(x, 0) = g_0(x), (u_0)_t(x, 0) = f_0(x).$$

For scattering states, we consider the following question: given data  $(g, f) \in \dot{H}^1 \times L^2$  and a corresponding solution  $u \in \dot{H}^1 \times L^2$  to the perturbed problem  $\square u = F(u, t)$  with initial data  $(g, f) \in \dot{H}^1 \times L^2$ , can we find data  $(g_0, f_0) \in \dot{H}^1 \times L^2$  such that the solution  $u_0 \in \dot{H}^1 \times L^2$  to the corresponding homogeneous problem  $\partial_{tt} u_0 - \Delta u_0 = 0$ ,  $(g_0, f_0) \in \dot{H}^1 \times L^2$  is such that

$$(7.6) \quad \|u(t) - u_0(t)\|_{\dot{H}^1 \times L^2} \rightarrow 0, \quad t \rightarrow \infty$$

To do this, as we discuss about projections, we reformulate the wave equation as a Hamiltonian system,

$$(7.7) \quad \partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 \\ F(u) \end{pmatrix}.$$

Setting

$$(7.8) \quad U := \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, H_F := \begin{pmatrix} -\Delta & 0 \\ 0 & 1 \end{pmatrix} \text{ and } F(U) := \begin{pmatrix} 0 \\ F(u, t) \end{pmatrix},$$

we can rewrite the free wave equation as

$$(7.9) \quad \dot{U}_0 - JH_F U_0 = 0,$$

$$(7.10) \quad U_0[0] = \begin{pmatrix} g_0 \\ f_0 \end{pmatrix}$$

and the perturbed wave equation as

$$(7.11) \quad \dot{U} - JH_F U = F(U),$$

$$(7.12) \quad U[0] = \begin{pmatrix} g \\ f \end{pmatrix}.$$

The solution of the free wave equation is given by

$$(7.13) \quad U_0 = e^{tJH_F} U_0[0],$$

on the other hand, by Duhamel's formula, the solution to the perturbed wave equation is given by

$$(7.14) \quad U[t] = e^{tJH_F} U[0] + \int_0^t e^{(t-s)JH_F} F(U(s)) \, ds.$$

We consider the charge transfer model,

$$(7.15) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x - vt)u = 0$$

for which

$$(7.16) \quad F(u, t) = -(V_1(x)u + V_2(x - vt)u)$$

**Theorem 7.1.** *Suppose  $u$  is a scattering state in the sense of Definition 1.2 which solves*

$$(7.17) \quad \partial_{tt} u - \Delta u + V_1(x)u + V_2(x - vt)u = 0.$$

*Write*

$$(7.18) \quad U = (u, u_t)^t \in C^0([0, \infty); \dot{H}^1) \times C^0([0, \infty); L^2),$$

*with initial data  $U[0] = (g, f)^t \in \dot{H}^1 \times L^2$ . Then there exist free data*

$$U_0[0] = (g_0, f_0)^t \in \dot{H}^1 \times L^2$$

*such that*

$$(7.19) \quad \|U[t] - e^{tJH_F} U_0[0]\|_{\dot{H}^1 \times L^2} \rightarrow 0$$

*as  $t \rightarrow \infty$ .*

*Proof.* We will still use the formulation in Theorem 5.2. We set  $A = \sqrt{-\Delta}$  and notice that

$$(7.20) \quad \|Af\|_{L^2} \simeq \|f\|_{\dot{H}^1}, \quad \forall f \in C^\infty(\mathbb{R}^3).$$

For real-valued  $u = (u_1, u_2) \in \mathcal{H} = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ , we write

$$(7.21) \quad U := Au_1 + iu_2.$$

We know

$$(7.22) \quad \|U\|_{L^2} \simeq \|(u_1, u_2)\|_{\mathcal{H}}.$$

We also notice that  $u$  solves (7.17) if and only if

$$(7.23) \quad U := Au + i\partial_t u$$

satisfies

$$(7.24) \quad i\partial_t U = AU + V_1 u + V_2(x - vt)u,$$

$$(7.25) \quad U(0) = Ag + if \in L^2(\mathbb{R}^3).$$

By Duhamel's formula, for fixed  $T$

$$(7.26) \quad U(T) = e^{iT A} U(0) - i \int_0^T e^{-i(T-s)A} (V_1 u(s) + V_2(\cdot - vs)u(s)) ds.$$

Applying the free evolution backwards, we obtain

$$(7.27) \quad e^{-iT A} U(T) = U(0) - i \int_0^T e^{is A} (V_1 u(s) + V_2(\cdot - vs)u(s)) ds.$$

Letting  $T$  go to  $\infty$ , we define

$$(7.28) \quad U_0(0) := U(0) - i \int_0^\infty e^{is A} (V_1 u(s) + V_2(\cdot - vs)u(s)) ds$$

By construction, we just need to show  $U_0[0]$  is well-defined in  $L^2$ , then automatically,

$$(7.29) \quad \|U(t) - e^{it A} U_0(0)\|_{L^2} \rightarrow 0.$$

It suffices to show

$$(7.30) \quad \int_0^\infty e^{is A} (V_1 u(s) + V_2(\cdot - vs)u(s)) ds \in L^2.$$

Then following the argument as in the proof of Theorem 5.2, we write  $V_1 = V_3 V_4$ ,  $V_2 = V_5 V_6$ .

We consider

$$(7.31) \quad \left\| \int_0^\infty e^{is A} V_3 V_4 u(s) ds \right\|_{L_x^2} \leq \|K_1\|_{L_{t,x}^2 \rightarrow L_x^2} \|V_4 u\|_{L_{t,x}^2},$$

where

$$(7.32) \quad (K_1 F)(t) := \int_0^\infty e^{is A} V_3 F(s) ds.$$

Similarly,

$$(7.33) \quad \left\| \int_0^\infty e^{is A} V_5 V_6(\cdot - vs)u(s) ds \right\|_{L_x^2} \leq \|\tilde{K}_1\|_{L_{t,x}^2 \rightarrow L_x^2} \|V_6(x - vt)u\|_{L_{t,x}^2},$$

where

$$(7.34) \quad (\tilde{K}_1 F)(t) := \int_0^\infty e^{is A} V_3(\cdot - vs)F(s) ds.$$

By the same argument in the proof of Theorem 5.4, one has

$$(7.35) \quad \|K_1\|_{L_{t,x}^2 \rightarrow L_x^2} \leq C_1, \quad \|\tilde{K}_1\|_{L_{t,x}^2 \rightarrow L_x^2} \leq C_2.$$



Therefore

$$(7.36) \quad \left\| \int_0^\infty e^{isA} V_3 V_4 u(s) ds \right\|_{L_x^2} \lesssim \|V_4 u\|_{L_{t,x}^2},$$

$$(7.37) \quad \left\| \int_0^\infty e^{isA} V_5 V_6(\cdot - vs) u(s) ds \right\|_{L_x^2} \lesssim \|V_6(x - vt)u\|_{L_{t,x}^2}.$$

By estimates (5.4) and (5.5) from Corollary 5.1,

$$(7.38) \quad \|V_4 u\|_{L_{t,x}^2} \lesssim \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^\alpha} |u(x, t)|^2 dx dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1},$$

$$(7.39) \quad \|V_6(x - vt)u\|_{L_{t,x}^2} \lesssim \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x - vt \rangle^\alpha} |u(x, t)|^2 dx dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

We conclude

$$\int_0^\infty e^{isA} (V_1 u(s) + V_2(\cdot - vs) u(s)) ds \in L^2$$

with

$$(7.40) \quad \left\| \int_0^\infty e^{isA} (V_1 u(s) + V_2(\cdot - vs) u(s)) ds \right\|_{L^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

So

$$(7.41) \quad U_0(0) := U(0) - i \int_0^\infty e^{isA} (V_1 u(s) + V_2(\cdot - vs) u(s)) ds$$

is well-defined in  $L^2$  and

$$(7.42) \quad \|U(t) - e^{itA} U_0(0)\|_{L^2} \rightarrow 0.$$

Define

$$(7.43) \quad (g_0, f_0) := (A^{-1} \Re U_0(0), \Im U_0(0)).$$

By construction, notice that

$$(7.44) \quad U[t] = (A^{-1} \Re U(t), \Im U(t))$$

and

$$(7.45) \quad \|U[t] - e^{tJH_F} U_0[0]\|_{\dot{H}^1 \times L^2} \rightarrow 0.$$

We are done. □

The above theorem confirms that scattering states indeed scatter to free waves.

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